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Research Article

Classification of Exact Solutions for Some Nonlinear Partial Differential Equations with Generalized Evolution

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We obtain the classification of exact solutions, including soliton, rational, and elliptic solutions, to the one-dimensional general improved Camassa Holm KP equation and KdV equation by the complete discrimination system for polynomial method. In discussion, we propose a more general trial equation method for nonlinear partial differential equations with generalized evolution.

1. Introduction

To construct exact solutions to nonlinear partial differential equations, some important methods have been defined such as Hirota method, tanh-coth method, the exponential function method, (G'/G)-expansion method, the trial equation method, [1–15]. There are a lot of nonlinear evolution equations that are integrated using the various mathematical methods. Soliton solutions, compactons, singular solitons, and other solutions have been found by using these approaches. These types of solutions are very important and appear in various areas of applied mathematics.

In Section 2, we give a new trial equation method for nonlinear evolution equations with higher-order nonlinearity. In Section 3, as applications, we obtain some exact solutions to two nonlinear partial diffeential equations such as the one-dimensional general improved Camassa Holm KP equation [16]:

$$(u_t + 2ku_x - u_{xxt} + au^n(u^n)_x)_x + u_{yy} = 0, \tag{1.1}$$

the dimensionless form of the generalized KdV equation [17]:

$$u_t + au^n u_x + bu^{2n} u_x + \delta u_{xxx} = 0. \tag{1.2}$$

In discussion, we propose a more general trial equation method.

2. The Extended Trial Equation Method

Step 1. For a given nonlinear partial differential equation,

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{2.1}$$

take the general wave transformation:

$$u(x_1, x_2, \ldots, x_N, t) = u(\eta), \qquad \eta = \lambda\left(\sum_{j=1}^{N} x_j - ct\right), \tag{2.2}$$

where $\lambda \neq 0$ and $c \neq 0$. Substituting (2.2) into (2.1) yields a nonlinear ordinary differential equation:

$$N(u, u', u'', \ldots) = 0. \tag{2.3}$$

Step 2. Take the finite series and trial equation as follows:

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{2.4}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \tag{2.5}$$

Using (2.4) and (2.5), we can write

$$\begin{aligned} (u')^2 &= \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right)^2, \\ u'' &= \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2}\right), \end{aligned} \tag{2.6}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting these relations into (2.3) yields an equation of polynomial $\Omega(\Gamma)$ of Γ:

$$\Omega(\Gamma) = \varrho_s \Gamma^s + \cdots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{2.7}$$

According to the balance principle, we can find a relation of θ, ϵ, and δ. We can compute some values of θ, ϵ, and δ.

Step 3. Let the coefficients of $\Omega(\Gamma)$ all be zero will yield an algebraic equations system:

$$\varrho_i = 0, \quad i = 0, \ldots, s. \tag{2.8}$$

Solving this system, we will determine the values of $\xi_0, \ldots, \xi_\theta$, $\zeta_0, \ldots, \zeta_\epsilon$, and $\tau_0, \ldots, \tau_\delta$.

Step 4. Reduce (2.5) to the elementary integral form:

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{2.9}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we solve (2.9) and obtain the exact solutions to (2.3). Furthermore, we can write the exact traveling wave solutions to (2.1), respectively.

3. Applications

Example 3.1 (Application to the Camassa Holm KP equation). In order to look for travelling wave solutions of (1.1), we make the transformation $u(x, y, t) = u(\eta)$, $\eta = m(x+y-ct)$, where m and c are arbitrary constants. Then, integrating this equation with respect to η twice and setting the integration constant to zero, we obtain

$$(2k + 1 - c)u + \frac{a}{2}u^{2n} + m^2 c u'' = 0. \tag{3.1}$$

We use the following transformation:

$$u = v^{1/(2n-1)}. \tag{3.2}$$

Equation (3.1) turns into the equation

$$m^2 c(2n-1)vv'' + 2m^2 c(1-n)v'^2 + (2k+1-c)(2n-1)^2 v^2 + \frac{a}{2}(2n-1)^2 v^3 = 0. \tag{3.3}$$

Substituting (2.6) into (3.3) and using balance principle yield

$$\theta = \epsilon + \delta + 2. \tag{3.4}$$

After this solution procedure, we obtain the results as follows.

Case 1. If we take $\epsilon = 0$, $\delta = 1$, and $\theta = 3$, then

$$\left(v'\right)^2 = \frac{(\tau_1)^2\left(\xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0\right)}{\zeta_0}, \tag{3.5}$$

where $\xi_3 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (2.8) yields

$$\begin{gathered}
\xi_1 = \frac{-a(1-2n)^2\zeta_0\tau_0^3 + m^2\xi_0\tau_1^2(2+4k+4n+8kn+3a\tau_0)}{m^2\tau_0\tau_1(q+a\tau_0)},\\
\xi_2 = \frac{-2a(1-2n)^2\zeta_0\tau_0^3 + m^2\tau_1^2(q+3a\tau_0)}{m^2\tau_0^2(q+a\tau_0)},\\
\xi_3 = \frac{a\tau_1\left(-(1-2n)^2\zeta_0\tau_0^2 + m^2\xi_0\tau_1^2\right)}{m^2\tau_0^2(q+a\tau_0)}, \quad \xi_0 = \xi_0,\ \zeta_0 = \zeta_0,\ \tau_0 = \tau_0,\ \tau_1 = \tau_1,\\
c = \frac{(1-2n)^2\zeta_0\tau_0^2(q+a\tau_0)}{(1+2n)\left((1-2n)^2\zeta_0\tau_0^2 - m\xi_0\tau_1^2\right)},
\end{gathered} \tag{3.6}$$

where $q = (1+2k)(1+2n)$. Substituting these results into (2.5) and (2.9), we have

$$\begin{aligned}
&\pm(\eta - \eta_0)\\
&= m\sqrt{A}\int \frac{d\Gamma}{\sqrt{\Gamma^3 + \dfrac{-2a(1-2n)^2\zeta_0\tau_0^3 + m^2\tau_1^2(q+3a\tau_0)}{a\tau_1\left(-(1-2n)^2\zeta_0\tau_0^2 + m^2\xi_0\tau_1^2\right)}\Gamma^2 + \mathcal{A}\Gamma + \dfrac{m^2\xi_0\tau_0^2(q+a\tau_0)}{a\tau_1\left(-(1-2n)^2\zeta_0\tau_0^2 + m^2\xi_0\tau_1^2\right)}}},
\end{aligned} \tag{3.7}$$

where $\mathcal{A}$ denotes by $((-a(1-2n)^2\zeta_0\tau_0^4+m^2\xi_0\tau_0\tau_1^2(2+4k+4n+8kn+3a\tau_0))/a\tau_1^2(-(1-2n)^2\zeta_0\tau_0^2+m^2\xi_0\tau_1^2)))$, and $A=\zeta_0\tau_0^2(q+a\tau_0)/a\tau_1(-(1-2n)^2\zeta_0\tau_0^2+m^2\xi_0\tau_1^2)$. Integrating (3.7), we obtain the solutions to the (1.1) as follows:

$$\pm(\eta-\eta_0)=-2m\sqrt{A}\frac{1}{\sqrt{\Gamma-\alpha_1}}, \tag{3.8}$$

$$\pm(\eta-\eta_0)=2m\sqrt{\frac{A}{\alpha_2-\alpha_1}}\arctan\sqrt{\frac{\Gamma-\alpha_2}{\alpha_2-\alpha_1}}, \quad \alpha_2>\alpha_1, \tag{3.9}$$

$$\pm(\eta-\eta_0)=m\sqrt{\frac{A}{\alpha_1-\alpha_2}}\ln\left|\frac{\sqrt{\Gamma-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{\Gamma-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|, \quad \alpha_1>\alpha_2, \tag{3.10}$$

$$\pm(\eta-\eta_0)=2m\sqrt{\frac{A}{\alpha_1-\alpha_3}}F(\varphi,l), \quad \alpha_1>\alpha_2>\alpha_3, \tag{3.11}$$

where

$$A=\frac{\zeta_0\tau_0^2(q+a\tau_0)}{a\tau_1\left(-(1-2n)^2\zeta_0\tau_0^2+m^2\xi_0\tau_1^2\right)}, \qquad F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \tag{3.12}$$

$$\varphi=\arcsin\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}}, \qquad l^2=\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}. \tag{3.13}$$

Also α_1, α_2, and α_3 are the roots of the polynomial equation

$$\Gamma^3+\frac{\xi_2}{\xi_3}\Gamma^2+\frac{\xi_1}{\xi_3}\Gamma+\frac{\xi_0}{\xi_3}=0. \tag{3.14}$$

Substituting the solutions (3.8)–(3.10) into (2.4) and (3.2), we have

$$\begin{aligned}
u(x,y,t)&=\left[\tau_0+\tau_1\alpha_1+\frac{4\tau_1A}{(x+y-\mathcal{B}t-\eta_0/m)^2}\right]^{1/(2n-1)},\\
u(x,y,t)&=\left[\tau_0+\tau_1\alpha_2+\tau_1(\alpha_1-\alpha_2)\tanh^2\left(\frac{1}{2}\sqrt{\frac{\alpha_1-\alpha_2}{A}}\left(x+y-\mathcal{B}t+\frac{\eta_0}{m}\right)\right)\right]^{1/(2n-1)},\\
u(x,y,t)&=\left[\tau_0+\tau_1\alpha_1+\tau_1(\alpha_1-\alpha_2)\operatorname{cosech}^2\left(\frac{1}{2}\sqrt{\frac{\alpha_1-\alpha_2}{A}}(x+y-\mathcal{B}t)\right)\right]^{1/(2n-1)},
\end{aligned} \tag{3.15}$$

where $\mathcal{B}$ denote by $((1-2n)^2\zeta_0\tau_0^2(q+a\tau_0)/(1+2n)((1-2n)^2\zeta_0\tau_0^2-m\xi_0\tau_1^2))$.

If we take $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$, then the solutions (3.15) can reduce to rational function solution:

$$u(x,y,t) = \left[\frac{2\sqrt{\widetilde{A}}}{x+y-\left((1-2n)^2\alpha_1^2(q-a\tau_1\alpha_1)/(1+2n)\left((1-2n)^2\zeta_0\alpha_1^2-m\xi_0\right)\right)t}\right]^{2/(2n-1)}, \tag{3.16}$$

1-soliton solution:

$$u(x,y,t) = \frac{A_1}{\cosh^{2/(2n-1)}\left[B(x+y-vt)\right]}, \tag{3.17}$$

and singular soliton solution:

$$u(x,y,t) = \frac{A_2}{\sinh^{2/(2n-1)}\left[B(x+y-vt)\right]}, \tag{3.18}$$

where $\widetilde{A} = A\tau_1$, $A_1 = (\tau_1(\alpha_2-\alpha_1))^{1/(2n-1)}$, $A_2 = (\tau_1(\alpha_1-\alpha_2))^{1/(2n-1)}$, $B = (1/2)\sqrt{(\alpha_1-\alpha_2)/A}$, and $v = (1-2n)^2\zeta_0\alpha_1^2(q-a\tau_1\alpha_1)/(1+2n)((1-2n)^2\zeta_0\alpha_1^2-m\xi_0)$. Here, A_1 and A_2 are the amplitudes of the solitons, while v is the velocity and B is the inverse width of the solitons. Thus, we can say that the solitons exist for $\tau_1 > 0$.

Case 2. If we take $\epsilon = 0$, $\delta = 2$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1+2\tau_2\Gamma)^2\left(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right)}{\zeta_0}, \tag{3.19}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (2.8) yields as follows.
Subcase 2.1. It holds that

$$\xi_0 = \xi_0, \quad \xi_1 = \frac{\xi_4\tau_1^3}{4\tau_2^3}+\frac{4\xi_0\tau_2}{\tau_1}, \quad \xi_2 = \frac{5\xi_4\tau_1^2}{4\tau_2^2}+\frac{4\xi_0\tau_2^2}{\tau_1^2}, \quad \xi_3 = \frac{2\xi_4\tau_1}{\tau_2}, \quad \xi_4 = \xi_4,\ \tau_1 = \tau_1,\ \tau_2 = \tau_2,$$

$$\zeta_0 = -\frac{m^2\left(-16a\xi_0\tau_2^4+\xi_4\tau_1^2\left(a\tau_1^2+4\tau_2 q\right)\right)}{a(1-2n)^2\tau_1^2\tau_2^2}, \qquad \tau_0 = \frac{\tau_1^2}{4\tau_2},$$

$$c = 1+2k+\frac{a\left(\xi_4\tau_1^4-16\xi_0\tau_2^4\right)}{4(1+2n)\xi_4\tau_1^2\tau_2}, \tag{3.20}$$

where $q = (1+2k)(1+2n)$. Substituting these results into (2.5) and (2.9), we get

$$\pm(\eta-\eta_0) = m\sqrt{\frac{\left(16a\xi_0\tau_2^4 - \xi_4\tau_1^4\left(a\tau_1^2+4q\tau_2\right)\right)}{a(1-2n)^2\xi_4\tau_1^2\tau_2^2}} \times \int \frac{d\Gamma}{\sqrt{\Gamma^4 + (2\tau_1/\tau_2)\Gamma^3 + \left(\left(5\tau_1^2/4\tau_2^2\right) + \left(4\xi_0\tau_2^2/\tau_1^2\right)\right)\Gamma^2 + \left(\left(\tau_1^3/4\tau_2^3\right) + \left(4\xi_0\tau_2/\tau_1\right)\right)\Gamma + \left(\xi_0/\xi_4\right)}}. \tag{3.21}$$

Integrating (3.21), we obtain the solutions to (1.1) as follows:

$$\pm(\eta-\eta_0) = -\frac{mB}{\Gamma-\alpha_1}, \tag{3.22}$$

$$\pm(\eta-\eta_0) = \frac{2mB}{\alpha_1-\alpha_2}\sqrt{\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}}, \quad \alpha_1 > \alpha_2, \tag{3.23}$$

$$\pm(\eta-\eta_0) = \frac{mB}{\alpha_1-\alpha_2}\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|, \tag{3.24}$$

$$\pm(\eta-\eta_0) = \frac{mB}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}-\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}+\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}\right|, \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{3.25}$$

$$\pm(\eta-\eta_0) = \frac{2mB}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{3.26}$$

where

$$B = \sqrt{\frac{\left(16a\xi_0\tau_2^4 - \xi_4\tau_1^4\left(a\tau_1^2+4q\tau_2\right)\right)}{a(1-2n)^2\xi_4\tau_1^2\tau_2^2}}, \qquad F(\varphi,l) = \int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \tag{3.27}$$

$$\varphi = \arcsin\sqrt{\frac{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}}, \qquad l^2 = \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{3.28}$$

Also α_1, α_2, α_3, and α_4 are the roots of the polynomial equation:

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4} = 0. \tag{3.29}$$

Substituting the solutions (3.22)–(3.25) into (2.4) and (3.2), we have

$$u(x,y,t) = \left[\tau_0 + \tau_1\alpha_1 \pm \frac{\tau_1 B}{\mathcal{C} - (\eta_0/m)} + \tau_2\left(\alpha_1 \pm \frac{B}{\mathcal{C} - (\eta_0/m)}\right)^2\right]^{1/(2n-1)},$$

$$u(x,y,t) = \left\{\tau_0 + \tau_1\alpha_1 + \frac{4B^2(\alpha_2-\alpha_1)\tau_1}{4B^2 - \left[(\alpha_1-\alpha_2)(\mathcal{C}-\eta_0)\right]^2} + \tau_2\left(\alpha_1 + \frac{4B^2(\alpha_2-\alpha_1)}{4B^2 - \left[(\alpha_1-\alpha_2)(\mathcal{C}-\eta_0)\right]^2}\right)^2\right\}^{1/(2n-1)},$$

$$u(x,y,t) = \left\{\tau_0 + \tau_1\alpha_2 + \frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right] - 1} + \tau_2\left(\alpha_2 + \frac{(\alpha_2-\alpha_1)}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right] - 1}\right)^2\right\}^{1/(2n-1)},$$

$$u(x,y,t) = \left\{\tau_0 + \tau_1\alpha_1 + \frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right] - 1} + \tau_2\left(\alpha_1 + \frac{(\alpha_1-\alpha_2)}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right] - 1}\right)^2\right\}^{1/(2n-1)},$$

$$u(x,y,t) = \left\{\tau_0 + \tau_1\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)(\mathcal{C})\right]} + \tau_2\left(\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)(\mathcal{C})\right]}\right)^2\right\}^{1/(2n-1)}, \tag{3.30}$$

where $\mathcal{C}$ denotes by $x + y - (1 + 2k + (a(\xi_4\tau_1^4 - 16\xi_0\tau_2^4)/4(1+2n)\xi_4\tau_1^2\tau_2))t$.

For simplicity, we can write the solutions (3.30) as follows:

$$u(x,y,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 \pm \frac{B}{\mathcal{C} - (\eta_0/m)}\right)^i\right]^{1/(2n-1)},$$

$$u(x,y,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 + \frac{4B^2(\alpha_2-\alpha_1)}{4B^2 - \left[(\alpha_1-\alpha_2)(\mathcal{C}-\eta_0)\right]^2}\right)^i\right]^{1/(2n-1)},$$

$$u(x,y,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_2 + \frac{(\alpha_2-\alpha_1)}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right]-1}\right)^i\right]^{1/(2n-1)},$$

$$u(x,y,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 + \frac{(\alpha_1-\alpha_2)}{\exp\left[((\alpha_1-\alpha_2)/B)(\mathcal{C}-\eta_0)\right]-1}\right)^i\right]^{1/(2n-1)},$$

$$u(x,y,t) = \left[\sum_{i=0}^{2}\tau_i\left(\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)(\mathcal{C})\right]}\right)^i\right]^{1/(2n-1)}, \tag{3.31}$$

Subcase 2.2. It holds that

$$\xi_0 = \xi_0, \quad \xi_1 = \xi_3 = 0, \quad \xi_2 = 2\sqrt{\xi_0\xi_4}, \quad \xi_4 = \xi_4, \quad \tau_1 = 0, \quad \tau_2 = \tau_2,$$
$$\zeta_0 = -\frac{4\left(qm^2\xi_4 + am^2\sqrt{\xi_0\xi_4}\tau_2\right)}{a(1-2n)^2\tau_2}, \qquad \tau_0 = \frac{\sqrt{\xi_0}\tau_2}{\sqrt{\xi_4}}, \qquad c = 1 + 2k + \frac{a\sqrt{\xi_0}\tau_2}{(1+2n)\sqrt{\xi_4}}, \tag{3.32}$$

where $q = (1+2k)(1+2n)$. Substituting these results into (2.5) and (2.9), we get

$$\pm(\eta-\eta_0) = m\sqrt{\frac{-4q\xi_4 - a\sqrt{\xi_0\xi_4}\tau_2}{a(1-2n)^2\xi_4\tau_2}}\int\frac{d\Gamma}{\sqrt{\Gamma^4 + \left(2\sqrt{\xi_0\xi_4}/\xi_4\right)\Gamma^2 + (\xi_0/\xi_4)}}. \tag{3.33}$$

Integrating (3.33), we obtain the solutions to the (1.1) as follows.

If we denote

$$F(\Gamma) = \Gamma^4 + \frac{2\sqrt{\xi_0\xi_4}}{\xi_4}\Gamma^2 + \frac{\xi_0}{\xi_4} = R^2 + d_1R + d_0, \tag{3.34}$$

where $\Gamma^2 = R$, $F(\Gamma) = G(R)$, then we can write complete discrimination system of $G(R)$ as follows:

$$\Delta = d_1^2 - 4d_0. \tag{3.35}$$

Correspondingly, there are the following two cases to be discussed.

(1) If $\Delta > 0$,then we have $F(\Gamma) = (\Gamma - \sqrt{\alpha_1})(\Gamma + \sqrt{\alpha_1})(\Gamma - \sqrt{\alpha_2})(\Gamma + \sqrt{\alpha_2})$, $\alpha_1 \neq \alpha_2$. Therefore, the solution is given by

$$\pm(\eta - \eta_0) = mC\sqrt{\alpha_1}F(\varphi, l), \tag{3.36}$$

where

$$C = \sqrt{\frac{-4q\xi_4 - a\sqrt{\xi_0\xi_4}\tau_2}{a(1-2n)^2\xi_4\tau_2}}, \qquad F(\varphi, l) = \int_0^{\varphi} \frac{d\psi}{\sqrt{1 - l^2\sin^2\psi}}, \tag{3.37}$$

$$\varphi = \arcsin\left(\frac{\Gamma}{\sqrt{\alpha_1}}\right), \quad l^2 = \frac{\alpha_1}{\alpha_2}, \quad \alpha_2 > \alpha_1. \tag{3.38}$$

(2) If $\Delta = 0$, then we have $F(\Gamma) = (\Gamma - \sqrt{\alpha_1})^2(\Gamma + \sqrt{\alpha_1})^2$. From here, the solutions can be found as

$$\pm(\eta - \eta_0) = \frac{mC}{\sqrt{\alpha_1}} \operatorname{arctanh}\left(\frac{\Gamma}{\sqrt{\alpha_1}}\right), \tag{3.39}$$

$$u(x,y,t) = \left\{\tau_0 + \tau_1\sqrt{\alpha_1}\tanh\left[\pm\frac{\sqrt{\alpha_1}}{C}\left(x + y - \left(1 + 2k + \frac{a\sqrt{\xi_0}\tau_2}{(1+2n)\sqrt{\xi_4}}\right)t - \frac{\eta_0}{m}\right)\right]\right.$$
$$\left. + \tau_2\alpha_1\tanh^2\left[\pm\frac{\sqrt{\alpha_1}}{C}\left(x + y - \left(1 + 2k + \frac{a\sqrt{\xi_0}\tau_2}{(1+2n)\sqrt{\xi_4}}\right)t - \frac{\eta_0}{m}\right)\right]\right\}^{1/(2n-1)}. \tag{3.40}$$

For simplicity, we can write (3.40) as follows:

$$u(x,y,t) = \left\{\sum_{i=0}^{2}\tau_i\left\{\sqrt{\alpha_1}\tanh\left[\pm\frac{\sqrt{\alpha_1}}{C}\left(x + y - \left(1 + 2k + \frac{a\sqrt{\xi_0}\tau_2}{(1+2n)\sqrt{\xi_4}}\right)t - \frac{\eta_0}{m}\right)\right]\right\}^i\right\}^{1/(2n-1)}. \tag{3.41}$$

Example 3.2 (Application to the generalized KdV equation). Using a complex variation η defined as $\eta = kx + wt$, we can convert (1.2) into ordinary different equation, which reads

$$wu' + aku^n u' + bku^{2n}u' + \delta k^3 u''' = 0, \tag{3.42}$$

where the prime denotes the derivative with respect to η. Integrating (3.42), and setting the constant of integration to be zero, we obtain

$$wu + \frac{ak}{n+1}u^{n+1} + \frac{bk}{2n+1}u^{2n+1} + \delta k^3 u'' = 0. \tag{3.43}$$

By the using of the transformation $u = v^{1/n}$, (3.43) reduces to

$$\delta k^3 n(n+1)(2n+1)vv'' + \delta k^3\left(1-n^2\right)(2n+1)\left(v'\right)^2 + bkn^2(n+1)v^4 + akn^2(2n+1)v^3 + n^2(n+1)(2n+1)wv^2 = 0. \tag{3.44}$$

Substituting (2.6) into (3.44) and using balance principle yield

$$\theta = \epsilon + 2\delta + 2. \tag{3.45}$$

If we take $\theta = 4$, $\epsilon = 0$, and $\delta = 1$, then

$$\left(v'\right)^2 = \frac{\tau_1^2\left(\xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0\right)}{\zeta_0}, \tag{3.46}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (2.8) yields

$$\xi_0 = \frac{\mathfrak{D}}{2\tau_1^3}, \qquad \xi_1 = \frac{\mathfrak{F}}{\tau_1^3},$$
$$\xi_2 = \xi_2, \quad \xi_3 = \xi_3, \quad \xi_4 = \frac{b(2+n)\xi_3 4\tau_1}{2(a+2an+2b(2+n)\tau_0)}, \quad \zeta_0 = -\frac{k^2(1+n)(2+n)(1+2n)\delta\xi_3}{2n^2(a+2an+2b(n+2)\tau_0)\tau_1},$$
$$\tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad w = \frac{-2k[3\xi_3\tau_0(a+2an+b(2+n)\tau_0) - \xi_2\tau_1(a+2an+b(2+n)\tau_0)]}{\xi_3(2+7n+7n^2+2n^3)}. \tag{3.47}$$

where $\mathfrak{D}$ denotes by $\tau_0^2(-(\xi_3\tau_0(4a(1+2n)+5b(2+n)\tau_0)/(a+2an+2b(2+n)\tau_0)) + 2\xi_2\tau_1)$ and $\mathfrak{F}$ denote by $\tau_0(-(\xi_3\tau_0(3a(1+2n)+4b(2+n)\tau_0)/(a+2an+2b(2+n)\tau_0)) + 2\xi_2\tau_1)$. Substituting these results into (2.5) and (2.9), we can write

$$\pm(\eta-\eta_0) = \frac{k}{\tau_1}\sqrt{\frac{-(1+n)(1+2n)\delta}{4n^2b}} \times \int \frac{d\Gamma}{\sqrt{\Gamma^4 + \frac{\mathcal{N}}{2b(2+n)\tau_1}\Gamma^3 + \frac{\xi_2(\mathcal{N})}{2b(2+n)\xi_2\tau_1}\Gamma^2 + \frac{\mathfrak{F}(\mathcal{N})}{2b(2+n)\xi_3\tau_1^4}\Gamma + \frac{\mathfrak{D}(\mathcal{N})}{4b(2+n)\xi_3\tau_1^4}}}, \tag{3.48}$$

where $\mathcal{N}$ denotes by $a + 2an + b(2 + n)\tau_0$. Integrating (3.48), we obtain the solutions to (1.2) as follows:

$$\pm(\eta - \eta_0) = -\frac{kB}{\Gamma - \alpha_1}, \tag{3.49}$$

$$\pm(\eta - \eta_0) = \frac{2kB}{\alpha_1 - \alpha_2}\sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \quad \alpha_2 > \alpha_1, \tag{3.50}$$

$$\pm(\eta - \eta_0) = \frac{kB}{\alpha_1 - \alpha_2}\ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|, \tag{3.51}$$

$$\pm(\eta - \eta_0) = \frac{kB}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}\ln\left|\frac{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} - \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} + \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}\right|, \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{3.52}$$

$$\pm(\eta - \eta_0) = \frac{2kB}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l), \qquad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{3.53}$$

where

$$B = \frac{1}{\tau_1}\sqrt{\frac{-(1 + n)(1 + 2n)\delta}{4n^2 b}}, \qquad F(\varphi, l) = \int_0^{\varphi}\frac{d\psi}{\sqrt{1 - l^2\sin^2\psi}}, \tag{3.54}$$

$$\varphi = \arcsin\sqrt{\frac{(\Gamma - \alpha_1)(\alpha_2 - \alpha_4)}{(\Gamma - \alpha_2)(\alpha_1 - \alpha_4)}}, \qquad l^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}. \tag{3.55}$$

Also α_1, α_2, α_3, and α_4 are the roots of the polynomial equation:

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4} = 0. \tag{3.56}$$

Substituting the solutions (3.49)–(3.52) into (2.4) and (3.2), we find

$$u(x, t) = \left[\tau_0 + \tau_1\alpha_1 \pm \frac{\tau_1 B}{\mathcal{M} - (\eta_0/k)}\right]^{1/n},$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{4B^2(\alpha_2-\alpha_1)\tau_1}{4B^2-\left[(\alpha_1-\alpha_2)\left(\mathcal{M}-\left(\eta_0/k\right)\right)\right]^2}\right\}^{1/n},$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\left((\alpha_1-\alpha_2)/B\right)\left(\mathcal{M}-\left(\eta_0/k\right)\right)\right]-1}\right\}^{1/n},$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\left((\alpha_1-\alpha_2)/B\right)\left(\mathcal{M}-\left(\eta_0/k\right)\right)\right]-1}\right\}^{1/n},$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)\mathcal{M}\right]}\right\}^{1/n},\tag{3.57}$$

where $\mathcal{M}$ denotes by $x+((-2[3\xi_3\tau_0(a+2an+b(2+n)\tau_0)-\xi_2\tau_1(a+2an+b(2+n)\tau_0)])/\xi_3(2+7n+7n^2+2n^3))t$.

If we take $\tau_0=-\tau_1\alpha_1$ and $\eta_0=0$, then the solutions (3.57) can reduce to rational function solutions:

$$u(x,t)=\left[\pm\frac{\widetilde{B}}{x+(2\tau_1(a+2an-b(2+n)\tau_1\alpha_1)(3\xi_3\alpha_1+\xi_2)/\xi_3(2+7n+7n^2+2n^3))t}\right]^{1/n},$$

$$u(x,t)=\left\{\frac{4\widetilde{B}^2(\alpha_2-\alpha_1)}{\tau_1\left[4B^2-((\alpha_1-\alpha_2)(x+(2\tau_1(a+2an-b(2+n)\tau_1\alpha_1)(3\xi_3\alpha_1+\xi_2)/\xi_3(2+7n+7n^2+2n^3))t))^2\right]}\right\}^{1/n},\tag{3.58}$$

traveling wave solutions:

$$u(x,t)=\left\{\frac{(\alpha_2-\alpha_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{(\alpha_1-\alpha_2)}{2B}\left(x+\frac{2\tau_1(a+2an-b(2+n)\tau_1\alpha_1)(3\xi_3\alpha_1+\xi_2)}{\xi_3(2+7n+7n^2+2n^3)}t\right)\right]\right\}\right\}^{1/n},\tag{3.59}$$

and soliton solution:

$$u(x,t)=\frac{A_3}{(D+\cosh[B_1(x-vt)])^{1/n}},\tag{3.60}$$

where $\widetilde{B}=B\tau_1$, $A_3=(2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1/(\alpha_3-\alpha_2))^{1/n}$, $B_1=\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B$, $D=(2\alpha_1-\alpha_2-\alpha_3)/(\alpha_3-\alpha_2)$, and $v=-2\tau_1(a+2an-b(2+n)\tau_1\alpha_1)(3\xi_3\alpha_1+\xi_2)/\xi_3(2+7n+7n^2+2n^3)$. Here, A_3 is the amplitude of the soliton, while v is the velocity and B_1 is the inverse width of the soliton. Thus, we can say that the solitons exist for $\tau_1<0$.

4. Discussion

Thus we give a more general extended trial equation method for nonlinear partial differential equations as follows.

Step 1. The extended trial equation (2.4) can be reduced to the following more general form:

$$u = \frac{A(\Gamma)}{B(\Gamma)} = \frac{\sum_{i=0}^{\delta} \tau_i \Gamma^i}{\sum_{j=0}^{\mu} \omega_j \Gamma^j}, \tag{4.1}$$

where

$$\left(\Gamma'\right)^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \tag{4.2}$$

Here, $\tau_i \ \ (i = 0, \ldots, \delta)$, $\omega_j \ \ (j = 0, \ldots, \mu)$, $\xi_\varsigma \ \ (\varsigma = 0, \ldots, \theta)$, and $\zeta_\sigma \ \ (\sigma = 0, \ldots, \epsilon)$ are the constants to be determined.

Step 2. Taking trial equations (4.1) and (4.2), we derive the following equations:

$$\begin{aligned} \left(u'\right)^2 &= \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \frac{\left(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma)\right)^2}{B^4(\Gamma)}, \\ u'' &= \frac{\left(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma)\right)\left\{\left(\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)\right)B(\Gamma) - 4\Phi(\Gamma)\Psi(\Gamma)B'(\Gamma)\right\}}{2B^3(\Gamma)\Psi^2(\Gamma)} \\ &\quad + \frac{2\Phi(\Gamma)\Psi(\Gamma)B(\Gamma)\left(A''(\Gamma)B(\Gamma) - A(\Gamma)B''(\Gamma)\right)}{2B^3(\Gamma)\Psi^2(\Gamma)}, \end{aligned} \tag{4.3}$$

and other derivation terms such as u''', and so on.

Step 3. Substituting u', u'' and other derivation terms into (2.3) yields the following equation:

$$\Omega(\Gamma) = \varrho_s \Gamma^s + \cdots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{4.4}$$

According to the balance principle, we can determine a relation of θ, ϵ, δ and μ.

Step 4. Letting the coefficients of $\Omega(\Gamma)$ all be zero will yield an algebraic equations system $\varrho_i = 0 \ \ (i = 0, \ldots, s)$. Solving this equations system, we will determine the values $\tau_0, \ldots \tau_\delta$; $\omega_0, \ldots, \omega_\mu$; $\xi_0 \ldots, \xi_\theta$, and $\zeta_0, \ldots, \zeta_\epsilon$.

Step 5. Substituting the results obtained in Step 4 into (4.2) and integrating equation (4.2), we can find the exact solutions of (2.1).

5. Conclusions and Remarks

In this study, we proposed a new trial equation method and used it to obtain some soliton and elliptic function solutions to the Camassa Holm KP equation and the one-dimensional general

improved KdV equation. Otherwise, we discussed a more general trial equation method. We think that the proposed method can also be applied to other nonlinear differential equations with nonlinear evolution.

Also, the convergence analysis of obtained elliptic solutions is given as follows:

$$F(\phi,l)=\int_0^{\phi}\frac{d\phi}{\sqrt{\left(1-l^2\sin^2\phi\right)}}, \tag{5.1}$$

where

$$\sin\phi=\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}},\qquad l=\sqrt{\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}}. \tag{5.2}$$

Especially, $\phi=\pi/2$, we have

$$F\left(\frac{\pi}{2},l\right)=\int_0^{\pi/2}\frac{d\phi}{\sqrt{\left(1-l^2\sin^2\phi\right)}}=\phi+\frac{1}{2}k^2v_2+\cdots+\frac{1.3\cdots(2n-1)}{2.4\cdots(2n)}k^{2n}v_{2n}+\cdots, \tag{5.3}$$

where $v_{2n}=\int_0^{\phi}\sin^{2n}\phi\,d\phi$. Taking the value $\phi=\pi/2$, we have $v_{2n}=((1.3\cdots(2n-1))/(2.4\cdots(2n)))(\pi/2)$. Therefore, if we take $\Gamma(t)=\alpha_2$ in (3.13), $\Gamma(t)=\alpha_4$ in (3.28) and (3.55), $\Gamma(t)=\sqrt{\alpha_1}$ in (3.38), for each t, then we have

$$F\left(\frac{\pi}{2},l\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2l^{2n}. \tag{5.4}$$

By the using radius of convergence of power series:

$$R=\frac{1}{\lim\limits_{n\to\infty}(a_{n+1}/a_n)}, \tag{5.5}$$

where $a_n=((2n)!/2^{2n}(n!)^2)^2$. We have the radius of convergence of power series $R=1$. We can say that power series converges for each $0<l<1$, diverges for each $l>1$. Consequently, the inequalities in (3.11), (3.26)–(3.53), and (3.38) are obtained by using $0<l<1$, respectively.

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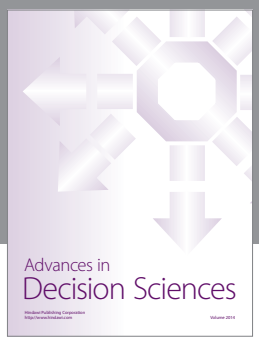
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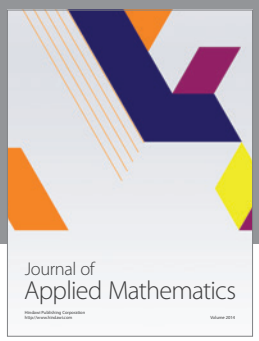
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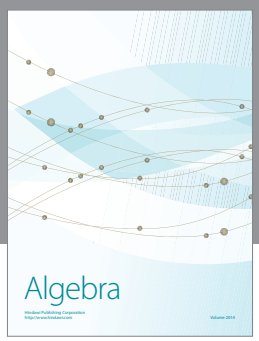
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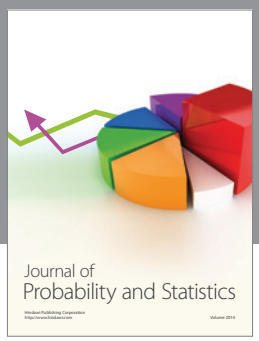
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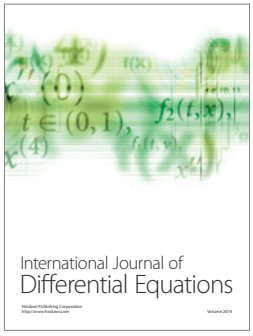


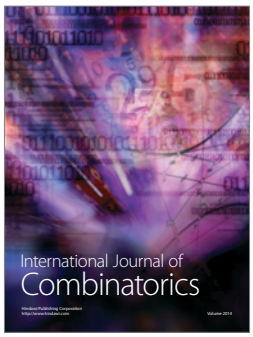


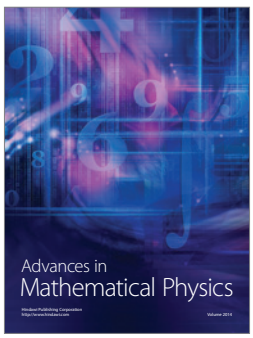


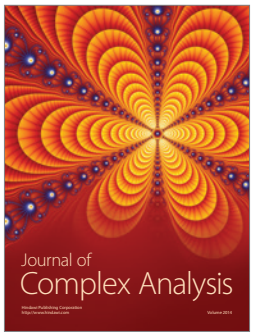




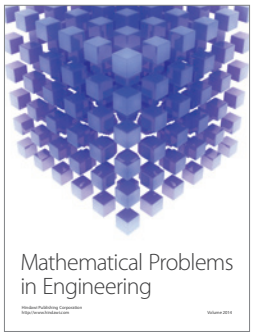


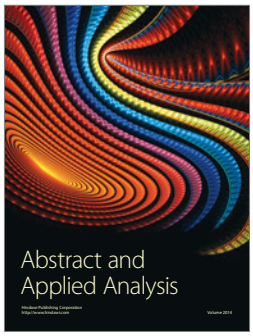


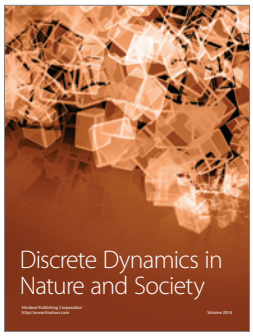


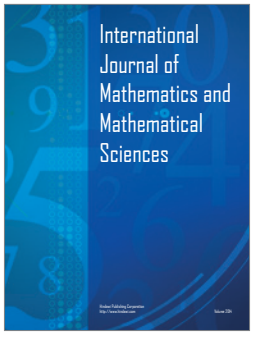






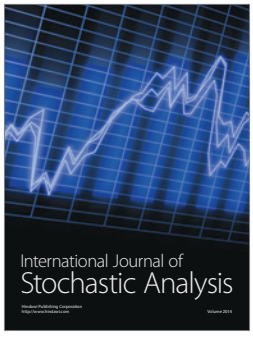














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