

FULL PAPER BOOK

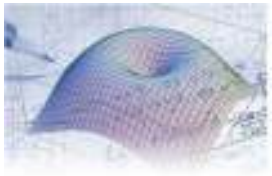
FIRST WESTERN BALKAN CONFERENCE ON MATHEMATICS AND APPLICATIONS

June 10-12, 2021, Prishtine, Kosova

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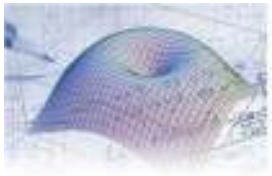
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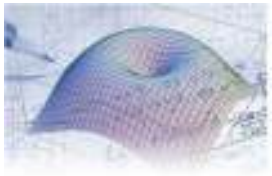
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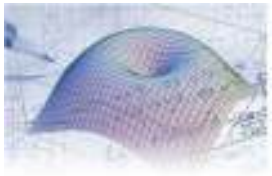
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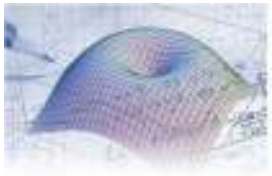
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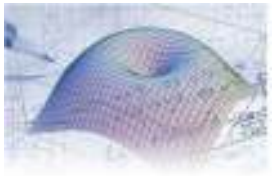


FOREWORD

On behalf of the Organizing Committee, we are very pleased to welcome you to the First Western Balkan Conference on Mathematics and Applications, June 10-12, 2021 (http://fwbcma2021.ilirias.com), in E-form
supported by Ilirias Research Institute(www.ilirias.com) and University of Prishtina (www.uni-pr.edu), Prishtina, Kosovo. We hope that, fwbcma2021 will be one of the most beneficial scientific events, bringing together mathematicians from all over the world, and demonstrating the vital role that mathematics play in any field of science and its application.

Naim L. Braha

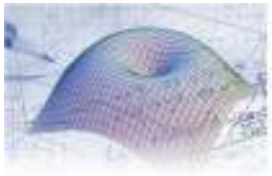
On behalf of the organizing committee



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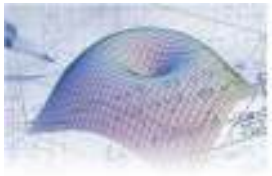
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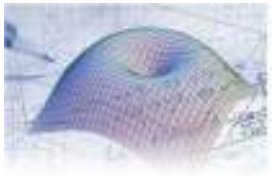
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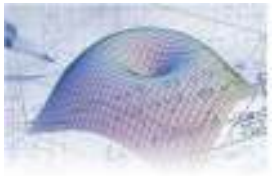


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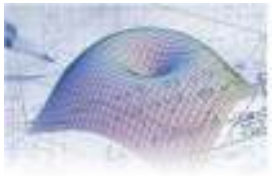
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The Transmission Eigenvalue Problem

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Non negative matrices and ideals

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Kellogg's theorem for diffeomophic minimisers of Dirichlet energy between doubly connected Riemann surfaces

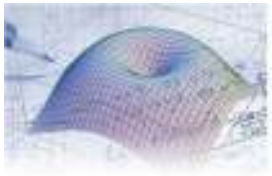
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Indispensable remarks on some recent results in metric fixed point theory

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Double Lacunary Statistical Convergence in Topological Groups II

Ali Farajzadeh

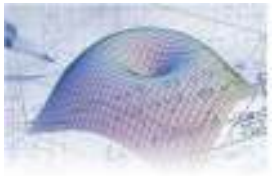
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On some generalized Banach contraction

Alexander G. Ramm

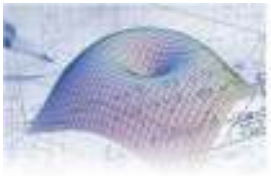
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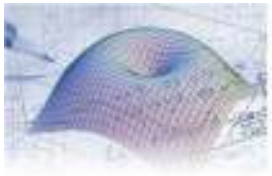


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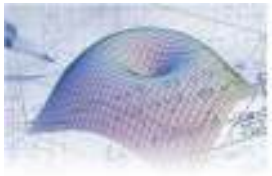


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A Surplus Application of the Interval Valued Triangular Fuzzy Numbers for the Linear Demand and Supply Functions

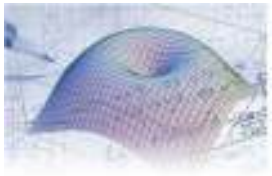
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Abstract: In this paper, we use the linear demand function $p = a - bx, 0 \leq x \leq a/b$ and linear supply function $p = e + gx, x \geq 0$,where a, b, e, g are positive constants and x is a quantity. We fuzzified the coefficients a, b, e, g by using the interval valued triangular fuzzy numbers in order to calculate the consumer surplus and producer surplus.

Keywords: Interval valued triangular fuzzy numbers; Consumer surplus and producer surplus.

1 Introduction

Many authors uses fuzzy set theory to study economic problems such as estimating optimal revenue [**2**] and optimal profit [**14**], calculating the consumer and producer surplus [**11,12**]. In [**12**], Yao and Wu considered linear demand function and linear supply function in which the demand quantity and the supply quantity are triangular fuzzy numbers. Then they calculated the consumer surplus and the producer surplus. Following paper, Wu [**11**] estimated these surpluses taking into consideration demand function and supply function to be linear or quadratic. He fuzzified



coefficients instead of quantity. In the both of the papers [**11,12**], triangular fuzzy numbers has been used for fuzzification.

Recently, Akbaş and Aytar [**1**] have calculated the consumer surplus of the demand function $p = a - bx, 0 \leq x \leq a/b$ and the producer surplus of the supply function $p = e + gx,\ x \geq 0$, where x is a quantity as an interval valued fuzzy number. In this paper, we fuzzify the coefficients $a, b, e,$ and g by using interval valued fuzzy numbers. Later we use the signed distance defuzzification method.

2 Preliminaries

The concept of interval valued fuzzy numbers was first defined by Zadeh [**15**]. Now we recall this concept using Gorzalczany's [**4**] notations as follows:

An interval valued fuzzy set (an IV fuzzy set) on the set of real numbers is given by $$A = \left\{\left(x, \left[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)\right]\right) : x \in R\right\} \quad (2.1)$$ $\underline{\mu}_{\tilde{A}}, \overline{\mu}_{\tilde{A}} : X \rightarrow [0,1]$; for all $x \in R, \underline{\mu}_{\tilde{A}}(x) \leq \overline{\mu}_{\tilde{A}}(x).$ (2.2)

Let

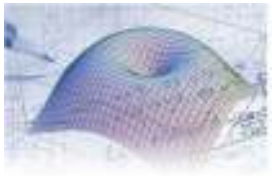
$$\tilde{\mu}_{\tilde{A}}(x) = \left[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)\right],\ x \in R$$

and let $D[0,1]$ denote the family of all closed intervals contained in the interval $[0,1]$, i.e.

$$\text{for all} a_1, a_2 \in [0,1], a_1 \leq a_2, [a_1, a_2] \in D[0,1].$$

Therefore, for each fixed $x \in R$, $\tilde{\mu}_{\tilde{A}}(x) \in D[0,1]$.Hence, expression (2.1) and (2.2) may be rewritten in a more concise form as

$$\tilde{\tilde{A}} = \left\{\left(x, \tilde{\mu}_{\tilde{A}}(x)\right) : \tilde{\mu}_{\tilde{A}} : X \rightarrow D[0,1], x \in R\right\}.$$



Thus, interval $\tilde{\mu}_{\tilde{\tilde{A}}}(x) = \left[\underline{\mu}_{\tilde{\tilde{A}}}(x), \overline{\mu}_{\tilde{\tilde{A}}}(x)\right]$ (a subset of $[0,1]$) and not a number from the interval $[0,1]$ is assigned to each element$x \in R$.It follows from definition that an arbitrary IVfuzzy set $\tilde{\tilde{A}}$ is unequivocally represented by the lower and upper limits $\underline{\mu}_{\tilde{\tilde{A}}}, \overline{\mu}_{\tilde{\tilde{A}}}$ respectively, of its generalized membership $\tilde{\mu}_{\tilde{\tilde{A}}}$.

If we take the fuzzy sets of $\underline{\mu}_{\tilde{\tilde{A}}}$ and $\overline{\mu}_{\tilde{\tilde{A}}}$ used here as a triangular fuzzy number, we get the definition of an interval valued triangular fuzzy number [**10**]. For the sake of convenience, we will use interval valued fuzzy number Chiao [**3**] uses as interval valued general triangular fuzzy number, that is if we choose the upper and lower membership functions as

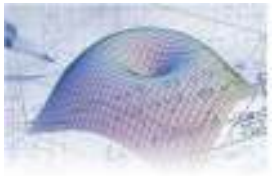
$$\overline{\mu}_{\tilde{\tilde{A}}}(y) = \begin{cases} \dfrac{y-a}{m-a} & , a \le y \le m \\ \dfrac{y-d}{m-d} & , m \le y \le d \\ 0 & , otherwise \end{cases}$$

$$\underline{\mu}_{\tilde{\tilde{A}}}(y) = \begin{cases} \dfrac{y-b}{m-b} & , b \le y \le m \\ \dfrac{y-c}{m-c} & , m \le y \le c \\ 0 & , otherwise \end{cases}$$

(where$a \le b \le m \le c \le d$) in (2.1) and (2.2) respectively, then $\tilde{\tilde{A}}$ is called an interval valued triangular (briefly, IVT) fuzzy number and is denoted as $\tilde{\tilde{A}} = ([a,b], m, [c,d])$. We denote the set of all IVT fuzzy numbers by$F(IVT)$[**3**].

Now let us briefly review the operations of *summation* and *scalar multiplication* on the set$F(IVT)$. For $\tilde{\tilde{x}}_1 = ([a,b], m_1, [c,d])$, $\tilde{\tilde{x}}_2 = ([e,f], m_2, [g,h]) \in F(\mathrm{T})$, summation, subtraction and scalar multiplication are defined as

$$\tilde{\tilde{x}}_1 + \tilde{\tilde{x}}_2 = ([a+e, b+f], m_1 + m_2, [c+g, d+h]),$$



$$\tilde{\tilde{x}}_1 - \tilde{\tilde{x}}_2 = \tilde{\tilde{x}}_1 + \left(-\tilde{\tilde{x}}_2\right) = ([a-h, b-g], m_1 - m_2, [c-f, d-e]),$$

$$k\tilde{\tilde{x}}_1 = k.\tilde{\tilde{x}}_1 = \begin{cases} [ka, kb], km_1, [kc, kd] & , k > 0 \\ [kd, kc], km_1, [kb, ka] & , k < 0 \end{cases}$$

The operations that enable converting an IV fuzzy numbers into an triangular fuzzy numbers and maintaining, at least partially, pieces of information stored in the former one, are frequently needed. This kind of operation is called type reduction by Karnik and Mendel [**5,6**] and Mendel [**7**]. In 2006 Niewiadomski et al. [**8**] have defined different approaches for this process as follows.

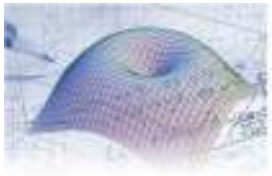
Let $\tilde{\tilde{A}}$ be an IVT fuzzy number and $\underline{\mu}_{\tilde{\tilde{A}}}$ and $\overline{\mu}_{\tilde{\tilde{A}}}$ be its lower and upper membership functions, respectively. The following operations transform $\tilde{\tilde{A}}$ into a triangular fuzzy number:

$$TI_I(\tilde{\tilde{A}}) = \overline{\mu}_{\tilde{\tilde{A}}}(y), y \in R \quad (2.3)$$

$$TI_K(\tilde{\tilde{A}}) = \underline{\mu}_{\tilde{\tilde{A}}}(y), y \in R \quad (2.4)$$

$$TI_O(\tilde{\tilde{A}}) = \frac{\underline{\mu}_{\tilde{\tilde{A}}}(y) + \overline{\mu}_{\tilde{\tilde{A}}}(y)}{2}, y \in R. \quad (2.5)$$

Here, lower indices mean: I – optimistic, K – pessimistic, O – realistic. Optimistic means that in the triangular fuzzy number which is the result of (2.3), membership degrees for y's, are taken as the highest (the most optimistic) degrees in $\tilde{\tilde{A}}$ and (2.4), (2.5) are defined accordingly. It is also possible to modify (2.5) with a weighted average of $\underline{\mu}_{\tilde{\tilde{A}}}(y)$ and $\overline{\mu}_{\tilde{\tilde{A}}}(y)$, if there are premises that one of membership levels could be more influential on a description of phenomenon than the other. In that case, (2.5) is in the form of



$$TI_{AO}\left(\tilde{\tilde{A}}\right) = w_1\underline{\mu}_{\tilde{\tilde{A}}}(y) + w_2\overline{\mu}_{\tilde{\tilde{A}}}(y), y \in R \quad (2.6)$$

provided that $w_1 + w_2 = 1$. Formula (2.6) is a generalization of the definitions above [**8**].

Yao and Wu [**13**] introduced the concept of *signed distance* in the fuzzy sense as follows: If $\tilde{A} \in F(T)$ then the signed distance of $\tilde{A} = (a_1, a_2, a_3)$ measure from as

$$d^*\left(\tilde{A}, 0_1\right) = \frac{1}{4}(a_1 + 2a_2 + a_3).$$

3 Results

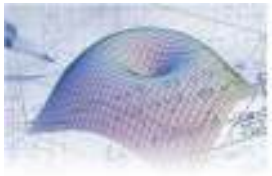
Let the demand function be $p = a - bx,\ 0 \le x \le \frac{a}{b}$ and the supply function be $p = e + gx,\ x \ge 0$ where $a > e > 0, b > 0, e > 0$ and $g > 0$are fixed numbers and p is the unit price with respect to quantity x. For a fixed unit price $p$we fuzzify the positive coefficients of the demand and of the supply functions as

$$\tilde{\tilde{a}} = ([a - \delta_{11}^a, a - \delta_{21}^a], a, [a + \delta_{22}^a, a + \delta_{12}^a]),$$
$$\tilde{\tilde{b}} = ([b - \delta_{11}^b, b - \delta_{21}^b], b, [b + \delta_{22}^b, b + \delta_{12}^b]),$$
$$\tilde{\tilde{e}} = ([e - \delta_{11}^e, e - \delta_{21}^e], e, [e + \delta_{22}^e, e + \delta_{12}^e]),$$
$$\tilde{\tilde{g}} = ([g - \delta_{11}^g, g - \delta_{21}^g], g, [g + \delta_{22}^g, g + \delta_{12}^g]),$$

where$0 \le \delta_{21}^t \le \delta_{11}^t < t, 0 \le \delta_{22}^t \le \delta_{12}^t$, and $t \in \{a, b, e, g\}$.

Hence the IVF demand function is

$$\tilde{\tilde{P}}_D = \begin{pmatrix} [a - \delta_{11}^a - (b + \delta_{12}^b)x, a - \delta_{21}^a - (b + \delta_{22}^b)x], \\ a - bx, \\ [a + \delta_{22}^a - (b - \delta_{21}^b)x, a + \delta_{12}^a - (b - \delta_{11}^b)x] \end{pmatrix}.$$



We have the triangular fuzzy numbers using the type reductions as

$$TI_{\dot{I}}(\tilde{\tilde{P}}_D) = (a - \delta_{11}^a - (b + \delta_{12}^b)x, a - bx, a + \delta_{12}^a - (b - \delta_{11}^b)x)$$

$$TI_K(\tilde{\tilde{P}}_D) = (a - \delta_{21}^a - (b + \delta_{22}^b)x, a - bx, a + \delta_{22}^a - (b - \delta_{21}^b)x)$$

$$TI_{AO}(\tilde{\tilde{P}}_D) = \begin{pmatrix} (1-w_1)(a - \delta_{11}^a - (b + \delta_{12}^b)x) + w_1(a - \delta_{21}^a - (b + \delta_{22}^b)x), \\ a - bx, \\ (1-w_1)(a + \delta_{12}^a - (b - \delta_{11}^b)x) + w_1(a + \delta_{22}^a - (b - \delta_{21}^b)x) \end{pmatrix}$$

$$TI_O(\tilde{\tilde{P}}_D) = \begin{pmatrix} \frac{a - \delta_{11}^a - (b + \delta_{12}^b)x + a - \delta_{21}^a - (b + \delta_{22}^b)x}{2}, \\ \frac{2(a - bx)}{2}, \\ \frac{a + \delta_{12}^a - (b - \delta_{11}^b)x + a + \delta_{22}^a - (b - \delta_{21}^b)x}{2} \end{pmatrix}.$$

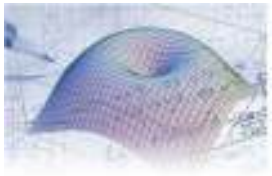
The signed distance of triangular fuzzy number $TI_O(\tilde{\tilde{P}}_D)$ is

$$E_1(x) = \frac{1}{4}\begin{pmatrix} \frac{a - \delta_{11}^a - (b + \delta_{12}^b)x + a - \delta_{21}^a - (b + \delta_{22}^b)x}{2} \\ + \frac{4(a - bx)}{2} \\ + \frac{a + \delta_{12}^a - (b - \delta_{11}^b)x + a + \delta_{22}^a - (b - \delta_{21}^b)x}{2} \end{pmatrix}$$

$$= a - bx + \frac{1}{8}\begin{pmatrix} (\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a) + \\ \left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right]x \end{pmatrix} \quad (3.1)$$

where E_1 estimates the demand quantity in the IV fuzzy sense when the price is p.

On the other hand, the IVF supply function is



$$\tilde{\tilde{P}}_S = \begin{pmatrix} \left[e - \delta_{11}^e + (g - \delta_{11}^g)x, e - \delta_{21}^e + (g - \delta_{21}^g)x\right], \\ e + gx, \\ \left[e + \delta_{22}^e + (g + \delta_{22}^g)x, e + \delta_{12}^e + (g + \delta_{12}^g)x\right] \end{pmatrix}.$$

We have the triangular fuzzy numbers using the type reductions as

$$TI_j(\tilde{\tilde{P}}_S) = (e - \delta_{11}^e + (g - \delta_{11}^g)x, e + gx, e + \delta_{12}^e + (g + \delta_{12}^g)x)$$

$$TI_K(\tilde{\tilde{P}}_S) = (e - \delta_{21}^e + (g - \delta_{21}^g)x, e + gx, e + \delta_{22}^e + (g + \delta_{22}^g)x)$$

$$TI_{AO}(\tilde{\tilde{P}}_S) = \begin{pmatrix} (1 - w_1)(e - \delta_{11}^e + (g - \delta_{11}^g)x) + w_1(e - \delta_{21}^e + (g - \delta_{21}^g)x), \\ e + gx, \\ (1 - w_1)(e + \delta_{12}^e + (g + \delta_{12}^g)x) + w_1(e + \delta_{22}^e + (g + \delta_{22}^g)x) \end{pmatrix}$$

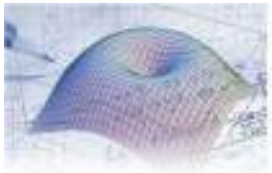
$$TI_O(\tilde{\tilde{P}}_S) = \begin{pmatrix} \frac{e - \delta_{11}^e + (g - \delta_{11}^g)x + e - \delta_{21}^e + (g - \delta_{21}^g)x}{2}, \\ \frac{2(e + gx)}{2}, \\ \frac{e + \delta_{12}^e + (g + \delta_{12}^g)x + e + \delta_{22}^e + (g + \delta_{22}^g)x}{2} \end{pmatrix}.$$

The signed distance of triangular fuzzy number $TI_O(\tilde{\tilde{P}}_S)$is

$$E_2(x) = \frac{1}{4}\begin{pmatrix} \frac{e - \delta_{11}^e + (g - \delta_{11}^g)x + e - \delta_{21}^e + (g - \delta_{21}^g)x}{2} \\ + \frac{4(e + gx)}{2} \\ + \frac{e + \delta_{12}^e + (g + \delta_{12}^g)x + e + \delta_{22}^e + (g + \delta_{22}^g)x}{2} \end{pmatrix}$$

$$= e + gx + \frac{1}{8}\begin{bmatrix} (\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e) + \\ \left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g)\right] x \end{bmatrix} \quad (3.2)$$

whereE_2 estimates the supply quantity in the IV fuzzy sense when the price is p.



Similar operations can be repeated easily by taking$TI_I(\tilde{\tilde{P}}_S)$or $TI_K(\tilde{\tilde{P}}_S)$ or $TI_{AO}(\tilde{\tilde{P}}_S)$ instead of $TI_O(\tilde{\tilde{P}}_S)$.

Now we recall the classical consumer surplus and producer surplus for the linear case as follows.

PROPOSITION 3.1 ([9]). If the demand equation is $p = a - bx$, and the supply function is $p = e + gx$ then the consumer surplus for the crisp case is

$$CSC = \frac{1}{2} b x_*^2 \tag{3.3}$$

and the producer surplus for the crisp case is

$$PSC = \frac{1}{2} g x_*^2 \tag{3.4}$$

where$x_*$is the equilibrium quantity.

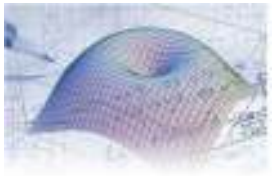
PROPOSITION 3.2 (CSF and PSF). If the demand equation is $\tilde{\tilde{P}}_D = \tilde{\tilde{a}} - \tilde{\tilde{b}}x$, and the supply function is$\tilde{\tilde{P}}_S = \tilde{\tilde{e}} + \tilde{\tilde{g}}x$then the consumer surplus for the interval valued fuzzy case is

$$CSF = x_* \left[a + \frac{1}{8}[(\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a)] - p_* \right] + \frac{x_*^2}{2} \left[\frac{1}{8} \left(\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b) \right] \right) - b \right]$$

and the producer surplus for the interval valued fuzzy case is

$$PSF = x_* \left[[p_* - e - \frac{1}{8}[(\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e)] \right] - \frac{x_*^2}{2} \left[\frac{1}{8} \left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g) \right] + g \right]$$

where



$$x_* = \frac{a - e + \frac{1}{8}[(\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a)] - \frac{1}{8}[(\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e)]}{b + g - \frac{1}{8}\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] + \frac{1}{8}\left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g)\right]}$$

and

$$p_* = a - bx_* + \frac{1}{8}\begin{pmatrix} (\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a) \\ +\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] x_* \end{pmatrix}.$$

PROOF. Let us calculate the equilibrium quantity. We have

$$a - bx + \frac{1}{8}\Big((\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a) + \left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] x\Big)$$

$$= e + gx + \frac{1}{8}\Big((\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e) + \left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g)\right]x\Big),$$

$$x = x_*$$

$$= \frac{a - e + \frac{1}{8}[(\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a)] - \frac{1}{8}[(\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e)]}{b + g - \frac{1}{8}\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] + \frac{1}{8}\left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g)\right]}$$

by using the equations (3.1) and (3.2). Then the equilibrium price can be determined by using E_1 or E_2 as:

$$p_* = E_1(x_*) = a - bx_* + \frac{1}{8}\begin{pmatrix} (\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a) \\ +\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] x_* \end{pmatrix}.$$

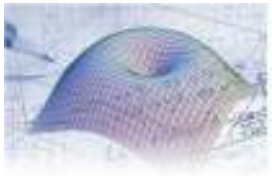
Hence we can calculate the consumer surplus:

$$CSF = \int_0^{x_*} [E_1(x) - p_*]dx$$

$$= \int_0^{x_*} \left[a - bx + \frac{1}{8}\begin{pmatrix} (\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a) \\ +\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right] x \end{pmatrix} - p_*\right] dx$$

$$= x_*\left[a + \frac{1}{8}[(\delta_{12}^a + \delta_{22}^a) - (\delta_{11}^a + \delta_{21}^a)] - p_*\right]$$

$$+ \frac{x_*^2}{2}\left[\frac{1}{8}\left(\left[(\delta_{11}^b + \delta_{21}^b) - (\delta_{12}^b + \delta_{22}^b)\right]\right) - b\right].$$



Similarly, we get the produces surplus:

$$PSF = \int_0^{x_*} [p_* - E_2(x)]dx$$
$$= \int_0^{x_*} \left[[p_* - e - gx - \frac{1}{8}\begin{pmatrix} (\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e) \\ +[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g)]x \end{pmatrix} \right] dx$$
$$= x_* \left[[p_* - e - \frac{1}{8}[(\delta_{12}^e + \delta_{22}^e) - (\delta_{11}^e + \delta_{21}^e)] \right]$$
$$- \frac{x_*^2}{2}\left[\frac{1}{8}\left[(\delta_{12}^g + \delta_{22}^g) - (\delta_{11}^g + \delta_{21}^g) \right] + g \right].$$

EXAMPLE 3.1. Define the demand function

$$p = 20 - 2x, 0 < x \leq 10$$

and the supply function

$$p = 10 + 2x, x \geq 0.$$

Here our coefficients are$a = 20, b = 2, e = 10$and $g = 2$.We use [**11,** Example 2.6] in order to compare our results and Yao and Wu's results [**11**].

Crisp Case([**11**]) From $20 - 2x = 10 + 2x$, the equilibrium quantity is $x_* = 2.5$ and equilibrium price is $p_* = 15$. Then we have the consumer surplus and producer surplus:

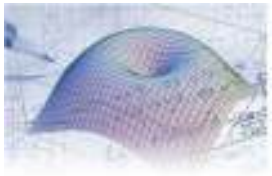
$$CSC = \frac{1}{2} b x_*^2 = 6.25,$$
$$PSC = \frac{1}{2} g x_*^2 = 6.25.$$

We note that in the rest of paper, if we require that the constants δ_{ij}^t to be compatible with Yao and Wu's [**11**] triangular fuzzy numbers.

Case 1 (*IVT*): Let the fuzzified coefficients

$$\begin{array}{cccc} \delta_{11}^a = 3 & \delta_{21}^a = 1 & \delta_{22}^a = 3 & \delta_{12}^a = 5 \\ \delta_{11}^b = 1.5 & \delta_{21}^b = 0.5 & \delta_{22}^b = 1 & \delta_{12}^b = 3 \\ \delta_{11}^e = 4 & \delta_{21}^e = 2 & \delta_{22}^e = 1 & \delta_{12}^e = 3 \\ \delta_{11}^g = 1.5 & \delta_{21}^g = 0.5 & \delta_{22}^g = 1 & \delta_{12}^g = 3. \end{array}$$

Hence the equilibrium quantity is
$x_* = 2.3889.$
Then the equilibrium price p_*can be calculated as



$$E_1(x_*) = E_2(x_*) = 15.1250.$$

The consumer surplus and producer surplus estimate as

$$CSF = 6.4202$$
$$PSF = 6.4202.$$

Case 2 (*IVT*):Let the fuzzified coefficients

$$\delta_{11}^a = 5 \quad \delta_{21}^a = 1 \quad \delta_{22}^a = 1 \quad \delta_{12}^a = 3$$
$$\delta_{11}^b = 1.5 \quad \delta_{21}^b = 0.5 \quad \delta_{22}^b = 0.5 \quad \delta_{12}^b = 1.5$$
$$\delta_{11}^e = 4 \quad \delta_{21}^e = 2 \quad \delta_{22}^e = 1 \quad \delta_{12}^e = 3$$
$$\delta_{11}^g = 1.5 \quad \delta_{21}^g = 0.5 \quad \delta_{22}^g = 1 \quad \delta_{12}^g = 3.$$

Hence the equilibrium quantity is

$$x_* = 2.3529.$$

Then the equilibrium price p_*can be calculated as

$$E_1(x_*) = E_2(x_*) = 15.0442.$$

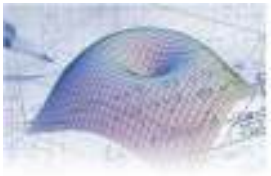
The consumer surplus and producer surplus estimate as

$$CSF = 6.4202,$$
$$PSF = 6.2285.$$

The values for the middle-type reduction methods using the new formulas calculated are given in Table 1.

	x_*	p_*	*PSF*	*CSF*
Crisp	2.5	15	6.25	6.25
Yao and Wu's [11], Case 1	2.357	15.167	6.481	6.481
Yao and Wu's[11], Case 2	2.231	14.871	5.807	4.977
Case 1 Middle type reduction	2.3839	15.1250	6.4202	6.4202
Case 2 Middle type reduction	2.3529	15.0442	6.4202	6.2285

Table 1.PSF/CSF for the triangular and *IVT*fuzzy numbers



As can be seen Case 2 in the Table 1, our total surplus value (12.6487) is higher than not only crisp value but also triangular value (Case 2). Moreover the surpluses can be increased if the type reduction technique is changed.

References

[1] Akbaş, B., Aytar, S., Consumer surplus and producer surplus of the linear demand and supplyfunctions using interval-valued triangular fuzzy numbers, International Journal of Applied Mathematics and Statistics, accepted.

[2] Chang, S.C. and Yao J.S. 2000. Optimal revenue for demand function in fuzzy sense, Fuzzy Sets and Systems, 111: 419-431.

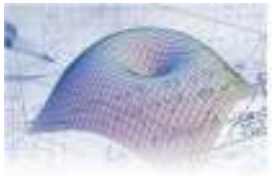
[3]Chiao, K. P. 2011 Multiple criteria group decision making with triangular interval type-2 fuzzy sets, In 2011 IEEE International Conference on Fuzzy Systems, 2575-2582.

[4] Gorzalczany, M.B. 1987. A method of inference in approximate reasoning based on interval valued fuzzy sets, Fuzzy Sets and Systems, 21: 1-17.

[5]Karnik, N.N. and Mendel, J.M. 1998. An Introduction to Type-2 Fuzzy Logic Systems, University of Southern California, Los Angeles.

[6]Karnik, N.N. and Mendel, J.M. 1999. Type-2 Fuzzy Logic Systems. IEEE Transactions on Fuzzy Systems 7(6): 643-658.

[7] Mendel, J.M. 2001, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, Prentice-Hall, Upper Saddle River, NJ.



[8]Niewiadomski, A. Ochelska, J. and Szczepaniak, P. S. 2006. Interval-valued linguistic summaries of databases, Control and Cybernetics, 35(2): 415-443.

[9]Varian, H.R. 2014. Intermediate Microeconomics with Calculus, W.W Norton & Company, New York.

[10]Wang, Y. J. 2019. Ranking triangular interval-valued fuzzy numbers based on the relative preference relation, Iranian Journal of Fuzzy Systems, 16(2): 123-136

[11]Wu, K., 1999, Consumer surplus and producer surplus in fuzzy sense, Fuzzy Sets and Systems, 103: 405-419.

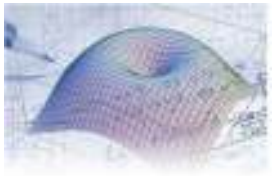
[12]Yao, J.S. and Wu, K. 1999. Consumer surplus and producer surplus for fuzzy demand and fuzzy supply, Fuzzy Sets and Systems, 103: 421-426.

[13]Yao, J.S. and Wu, K. 2000. Ranking fuzzy numbers based on decomposition principle and signed distance, Fuzzy Sets and Systems, 116: 275-288.

[14]Yao, J.S. and Chang, S.C. 2001. Economic principle on profit in fuzzy sense, Fuzzy Sets and Systems, ***117:*** *113-127.*

[15]Zadeh, L.A. 1975. The concept of a linguistic variable and its application to approximate reasoning. Information Sciences 8(3): 199-249.

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



On Bernstein Operators Based on Shape Parameter λ

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Abstract: In order to approximate functions many Bernstein type operators have been constructed using shape λ, where $\lambda \in [-1,1]$. In this work, we combine certain results about λ Bernstein type operators and related approximation properties.

Keywords: Convergence by operators, shape parameter λ convergence, approximation properties.

1. Introduction

A famous mathematician S. N. Bernstein [1] constructed polynomials nowadays called Bernstein polynomials which is familiar and widely-investigated polynomials in theory of approximation. Bernstein gave a simple and very elegant way to obtain Weierstrass approximation theorem with the help of his newly-constructed polynomials. For any continuous function $f(x)$ defined on $C[0,1]$, Bernstein polynomials of order n are given by

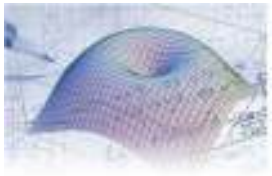
$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) b_{n,i}(x) \qquad (x \in [0,1]),$$

where the Bernstein basis functions $b_{n,i}(x)$ are defined by

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (i = 0, \dots, n).$$

Stancu [2] presented a generalization of Bernstein polynomials with the help of two parameters α and β such that $0 \leq \alpha \leq \beta$, as follows:

$$S_{n,\alpha,\beta}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right)\binom{n}{i} x^i (1-x)^{n-i} \qquad (x \in [0,1]).$$



If we take both the parameters $\alpha = \beta = 0$, then we get the classical Bernstein polynomials. The operators defined above are called Bernstein-Stancu operators.

In the next section we give a list of certain λ −Bersntein operators. In order to construct those operators we consider folowing newly defined Bézier bases [7]:

$$\tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x),$$

$$\tilde{b}_{n,i}(\lambda;x) = b_{n,i}(x) + \frac{n-2i+1}{n^2-1}\lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}\lambda b_{n+1,i+1}(x),$$

$$i = 1,2 \dots, n-1,$$

$$\tilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x),$$

in this case $\lambda \in [-1,1]$ and $b_{n,i}(x)$ are the Bernstein basis functions. By taking above operators into their account, they established various approximation results, namely, Korovkin and Voronovskaja-type theorems, rate of convergence via Lipschitz continuous functions, local approximation and other related results.

In the same year, Cai [8] generalized λ-Bernstein operators by constructing the Kantorovich-type λ-Bernstein operators as well as its Bézier variant and studied several approximation results.

Later, various approximation properties and asymptotic type results of the Kantorovich-type λ-Bernstein operators have been studied by Acu et al. [9].

Recently, Özger [10] obtained statistical approximation for λ-Bernstein operators including a Voronovskaja-type theorem in statistical sense. In the same article, he also constructed bivariate λ-Bernstein operators and studied their approximation properties.

The Bernstein operators are one of the most studied positive linear operators which were modified by many authors, we are mentioning some of them [3, 4, 11-18, 27].

2. Results on λ-Bernstein type operators

We are now ready to mention newly defined λ –Bernstein type operators:

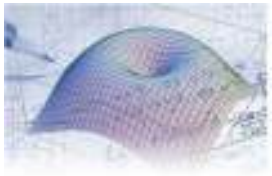
1. λ −Bersntein operators:

$$B_n^{\lambda}(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \tilde{b}_{n,i}(\lambda;x) \quad \text{Cai et al. [6].}$$

2. λ −Bersntein-Stancu operators:

$$B_{n,\alpha,\beta}^{\lambda}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right) \tilde{b}_{n,i}(\lambda;x) \quad \text{Srivastava et al. [23].}$$

3. λ −Bersntein-Schurer operators:



$$B_{n,\alpha,\beta}^{\lambda,d}(f;x)=\sum_{i=0}^{n+d} f\left(\frac{i}{n}\right)\tilde{s}_{n,i}(\lambda;x)$$ Özger [24].

4. $\lambda-$Bersntein-Kantorovich operators:

$$K_{n,\lambda}(h;x)=(n+1)\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(t)dt$$ Acu et al. [9].

5. $\lambda-$Bersntein-Schurer-Kantorovich operators:

$$K_{n,\lambda,a}(h;x)=(n+1)\sum_{i=0}^{n+a}\tilde{s}_{n,i}(\lambda;x)\int_{\frac{i}{n+a+1}}^{\frac{i+1}{n+a+1}} h(t)dt$$ Özger et al. [25].

6. $\lambda-$Bersntein-Kantorovich operators (with shifted knots):

$$K_{n,\lambda}^{(\alpha,\beta)}(f;x)=\left(\frac{n+\beta}{n}\right)^{n}(n+\beta+1)\sum_{i=0}^{n}\hat{b}_{n,i}^{(\alpha,\beta)}(\lambda;x)\int_{\frac{i+\alpha}{n+\beta+1}}^{\frac{i+\alpha+1}{n+\beta+1}} f(t)dt$$ Rahman et al. [26].

7. $\lambda-$ genuine Bernstein-Durrmeyer:

$$U_{n,\lambda}^{\rho}(f;x)=\sum_{i=1}^{n-1}\int_{0}^{1}\left[\frac{t^{i\rho-1}(1-t)^{(n-i)\rho-1}}{B(i\rho,(n-i)\rho)}f(t)dt\right]\tilde{b}_{n,i}(\lambda;x)+h(0)\tilde{b}_{n,0}(\lambda;x)+h(1)\tilde{b}_{n,n}(\lambda;x)$$ Acu et al. [21].

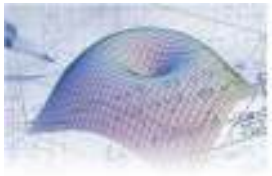
8. $\lambda-$ Szász $-$ Mirakian operators:

$$M_{n,\lambda}(h;x)=\sum_{i=0}^{n}\widetilde{m}_{n,i}(\lambda;x)f\left(\frac{i}{n}\right),\qquad x\in[0,\infty),$$ Qi et al. [20]

where

$$\widetilde{m}_{n,0}(\lambda;x)=m_{n,0}(x)-\frac{\lambda}{n+\alpha+1}m_{n+1,1}(x),$$
$$\widetilde{m}_{n,i}(\lambda;x)=m_{n,i}(x)+\frac{\lambda}{n^2-1}\begin{bmatrix}(n-2i+1)m_{n+1,i}(x)\\-(n-2i-1)m_{n+1,i+1}(z)\end{bmatrix}$$
$$\widetilde{m}_{n,n}(\lambda;x)=m_{n,n}(x)-\frac{\lambda}{n+1}m_{n+1,n}(x)$$

and



$$m_{n,,i}(z) = e^{-nx}\frac{(nx)^i}{i!}.$$

9. λ −Bersntein- Bézier -Kantorovich operators:

$$L_{n,\lambda,\beta}(h;z) = (n+1)\sum_{i=0}^{n} Q_{n,i}^{(\beta)}(\lambda;x)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)dt, \quad \text{Acu et al. [8]}$$

where

$$Q_{n,i}^{(\beta)}(\lambda;x) = \left[J_{n,i}^{(\beta)}(\lambda;x)\right]^{\beta} - \left[J_{n,i+1}^{(\beta)}(\lambda;x)\right]^{\beta} \quad \text{and}$$

$$J_{n,i}^{(\beta)}(\lambda;x) = \sum_{j=i}^{n} \tilde{b}_{n,j}(\lambda;x).$$

Following theorem is called Korovkin type theorem.

Theorem 1. λ-Bernstein type operators have the following

$$\lim_{n\to\infty} L_n^{\lambda}(f;x) = f(x)$$

uniformly in $[0,1]$, where $L_n^{\lambda} = B_n^{\lambda}, B_{n,\alpha,\beta}^{\lambda}, B_{n,\alpha,\beta}^{\lambda,d}, K_{n,\lambda}, K_{n,\lambda,a}, U_{n,\lambda}^{\rho}, M_{n,\lambda}, L_{n,\lambda,\beta}, K_{n,\lambda}^{(\alpha,\beta)}$.

Proof. Since λ -Bernstein type operators have the following relations

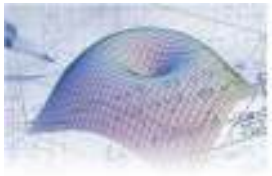
$L_{n,}^{\lambda}(1;x) = 1$ as $n \to \infty$;

$L_{n,}^{\lambda}(t;x) = x$ as $n \to \infty$;

$L_{n,}^{\lambda}(t^2;x) = x^2$ as $n \to \infty$

for $x \in [0,1]$, theorem is proved.

As an application of this theorem we consider the function $f(x) = \cos(x^3)$ and take $\lambda = 0.5$ and $n = 20$, and we provide convergence of λ −Bersntein operators in Figure 1.



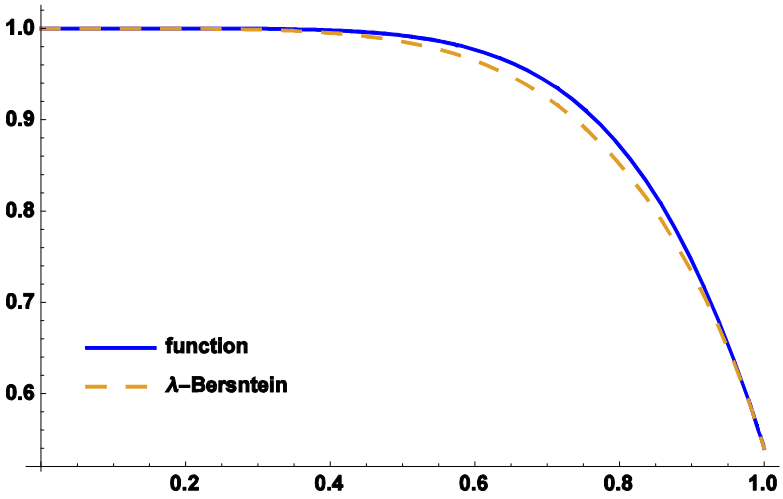


Figure 1. Convergence of λ −Bersntein operators

In all these studies, many approximation theorems including Korovkin type theorems, rates of convergence, point-wise convergence, shape preserving properties, global approximation formula, local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters, statistical convergence, weighteded statistical convergence, Grüss-Voronovskaja and a quantitative Voronovskaja types. Behavior of polinomials $\tilde{s}$ is considered in [25]. Also λ −Bersntein type operators are considered in weighted spaces.

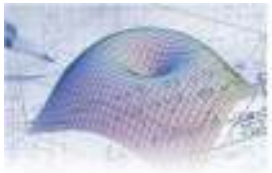
With the help of the provided numerical evaluations, certain tables and graphs were provied for λ −Bersntein type operators in order to illustrate the efficiency of related results when it is compared with other results in the literature.

Moreover, bivariate versions of λ −Bersntein type operators are constructed in certain papers (for instance see, [10]).

References:

[1] Bernstein, S.N. Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. Comm. Soc. Math. Kharkow 1913, 13(2), 1-2.

[2] Stancu, D.D. Asupra unei generalizari a polinoamelor lui Bernstein. Studia Univ. Babes-Bolyai, Ser. Math.-Phys. 1969, 14, 31-45.



[3] Acar, T.; Mohiuddine, S.A. and Mursaleen, M. Approximation by (p,q)-Baskakov-Durrmeyer-Stancu operators. Complex Anal. Oper. Theory 2018, 12, 1453-1468.

[4] Baxhaku, B.; Agrawal, P.N. Degree of approximation for bivariate extension of Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators. Appl. Math. Comput. 2017, 306(1), 56-72.

[5] Cai, Q.-B.; Lian, B.-Y.; Zhou, G. Approximation properties of λ-Bernstein operators, J. Inequal. Appl. 2018, 2018:61.

[6] Milovanovic G.V., Mursaleen M., Nasiruzzaman M, Modified Stancu type

Dunkl generalization of Szász-Kantorovich operators, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM 112(1) (2018) 135-151.

[7] Ye, Z.; Long, X.; Zeng, X.-M. Adjustment algorithms for Bézier curve and surface. International Conference on Computer Science and Education 2010, 1712-1716.

[8] Cai, Q.-B. The Bézier variant of Kantorovich type λ-Bernstein operators. J. Inequal. Appl. 2018, 2018:90.

[9] Acu, A. M.; Manav, N.; Sofonea, D. F. Approximation properties of λ-Kantorovich operators. J. Inequal. Appl. 2018, 2018:202.

[10] Özger F, Applications of Generalized Weighted Statistical Convergence to Approximation Theorems for Functions of One and Two Variables. Numerical Functional Analysis and Optimization, 2020, 41.16: 1990-2006.

[11] Mursaleen M., Ansari, K J., Khan, A, Some approximation results by (p,q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput. 264 (2015) 392-402.

[12] Büyükyazıcı İ., İbikli E., The properties of generalized Bernstein polynomials of two variables. Appl. Math. Comput. 2004, 156, 367-380.

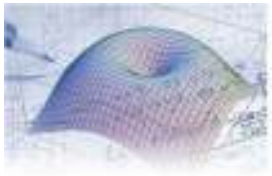
[13] Kajla A. Acar T., Blending type approximation by generalized Bernstein-Durrmeyer type operators, Miskolc Math Notes 19 (2018) 319-336.

[14] Mohiuddine S.A., Acar T., Alotaibi A., Construction of a new family of Bernstein-Kantorovich operators, Math. Meth. Appl. Sci. 40 (2017) 7749-7759.

[15] Acu A.M., Muraru, C., Approximation properties of bivariate extension of q-Bernstein-Schurer-Kantorovich operators. Results Math. 2015, 67(3), 265-279.

[16] Mishra V.N., Patel, P., On generalized integral Bernstein operators based on q-integers. Appl. Math. Comput. 2014, 242, 931-944.

[17] Mohiuddine S.A., Alamri B.A.S., Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 113(3) (2019) 1955-1973.



[18] Mishra V. N., Patel P., On generalized integral Bernstein operators based on q-integers, Appl. Math. Comput. 242 (2014) 931-944.

[19] Mohiuddine, S.A., Özger F., Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM (2020) 114:70.

[20] Qi Q, Guo D, Yang G., Approximation Properties of λ-Szász-Mirakian Operators. International Journal of Engineering Research and Technology. 2019; 12(5):662-669.

[21] Acu A M, Acar T, Radu V A., Approximation by modified $U_{n,\lambda}^{\rho}$ operators. RACSAM 2019; 113:2715-2729.

[22] Özger F., Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators, Filomat 33(11) (2019) 1-15.

[23] Srivastava H.M., Özger F., Mohiuddine S.A., Construction of Stancu-type Bernstein operators Based on Bézier bases with shape parameter λ, Symmetry 11(3) (2019), Article 316.

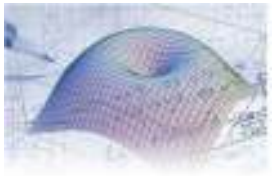
[24] Özger F., On new Bézier bases with Schurer polynomials and corresponding results in approximation theory, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat, 69(1) (2020) 376-393.

[25] Özger F., Demirci K., Yıldız S., Approximation by Kantorovich variant of λ-Schurer operators and related numerical results. In: Topics in Contemporary Mathematical Analysis and Applications, pp. 77–94. CRC Press, Boca Raton (2020). ISBN 9780367532666

[26] Rahman S, Mursaleen M, Acu AM., Approximation properties of λ-Bernsteinâ€ Kantorovich operators with shifted knots. Math Meth Appl Sci. 2019; 42(11):042-4053.

[27] Alotaibi, A., Özger, F., Mohiuddine, S.A. et al. Approximation of functions by a class of Durrmeyer–Stancu type operators which includes Euler's beta function. Adv Differ Equ 2021, 13 (2021). https://doi.org/10.1186/s13662-020-03164-0

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



A Note on Modified Beta-Type Polynomials

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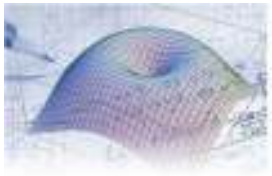
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Abstract: The purpose of this paper is to construct generating function for modified beta polynomials. By utilizing this generating function, we consider the modified beta-type polynomials and also investigate some basic properties of these polynomials. Moreover, we provide several fundamental equations and partial differential equations (PDEs) associated with this generating function.

Keywords: Beta-type polynomials, modified exponential function, generating functions, special polynomials

Introduction

Along this paper, the familiar notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are, respectively, referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers, and the set of all non-negative integers.
Polynomials are often used in both mathematics and engineering applications because they are closed under addition, multiplication, differentiation, integration and composition. Therefore, they have become an efficient tool to represent different systems, simulate various processes and compute final decisions. Recently, generating functions for special polynomials and numbers have also been used in many branches of mathematics, physics and engineering. Using the generating functions, many fundamental properties and identities for special polynomials and numbers can be obtained.



For $n,k \in \mathbb{N}_0$, the beta-type rational functions $\mathfrak{M}_{n,k}(x)$ are defined by the following formula (*cf.* [1-4]):

$$\mathfrak{M}_{n,k}(x) = (1+x)^{n-k} x^k.$$

Let $n,k \in \mathbb{N}_0$, the beta polynomials $\mathfrak{B}_{n,k}(x)$ of degree n on the interval $[-1,0]$ are defined as follows:

$$\mathfrak{B}_{n,k}(x) = (1+x)^{n-k} x^k,$$

where $k = 0,1,2,\ldots n.$

Simsek [1] constructed generating functions for the beta-type rational functions and the beta polynomials and then investigate some properties of these generating functions. In order to give derivative formulas, a recurrence relation and a variety of identities related to the beta-type rational functions and the beta polynomials, many functional equations and PDEs for these generating functions are derived in [1]. Also, some of these functional equations and PDEs are given.

The generating functions of the beta-type rational functions $\mathfrak{M}_{n,k}(x)$ are given by (cf. [1-4])

$$\mathfrak{b}_k(t,x) = \left(\frac{x}{1+x}\right)^k e^{t(1+x)} = \sum_{n=0}^{\infty} \mathfrak{M}_{n,k}(x) \frac{t^n}{n!},$$

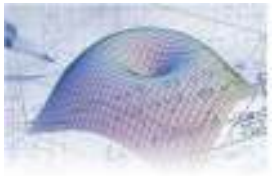
where $k \in \mathbb{N}_0$ and $t \in \mathbb{C}$. By (1.3), the generating functions of the beta polynomials $\mathfrak{B}_{n,k}(x)$ are provided by

$$\mathfrak{F}_k(t,x) = \mathfrak{b}_k(t,x) - \sum_{n=0}^{k} \frac{x^k}{(1+x)^{k-n}} \frac{t^n}{n!} = \sum_{n=k}^{\infty} \mathfrak{B}_{n,k}(x) \frac{t^n}{n!}.$$

In this paper, we construct generating function for modified beta polynomials. By utilizing this generating function, we consider the modified beta polynomials and also investigate some basic properties of these polynomials. Moreover, we provide several fundamental equations and partial differential equations (PDEs) associated with this generating function.

Modified Beta-Type Polynomials

In [5,6], the modified version of the Sumudu and Laplace integral transforms are defined and some of their properties are investigated. In the mentioned studies, some



advantages of modified forms are also examined. In this section, we introduce the modified beta polynomials and derive some properties of these polynomials.

Definition 1 *We introduce the modified beta polynomials* $\mathfrak{B}_{n,k}(x;a)$ *of degree* n *on the interval* $[-1,0]$ *as follows:*

$$\mathfrak{B}_{n,k}(x;a)=(\ln a)^n(1+x)^{n-k}x^k,$$

where $k=0,1,2,\ldots n.$

We note that when $a=e$, the modified beta polynomials $\mathfrak{B}_{n,k}(x;a)$ reduces to the beta polynomials $\mathfrak{B}_{n,k}(x)$ in (1.2).

Remark 1 *For mathematical convenience, we set*

$$\mathfrak{B}_{n,k}(x;a)=0$$

if $k>n$ or $k<0$.

We now perform to derive some properties of the modified beta polynomials.

By (2.1), we compute the first few modified beta polynomials as follows:

$$\mathfrak{B}_{0,0}(x;a)=1,\mathfrak{B}_{1,0}(x;a)=(1+x)\ln a,\mathfrak{B}_{1,1}(x;a)=x\ln a$$

$$\mathfrak{B}_{2,0}(x;a)=(\ln a)^2(1+x)^2,\mathfrak{B}_{2,1}(x;a)=(x+x^2)(\ln a)^2$$

$$\mathfrak{B}_{2,2}(x;a)=(x\ln a)^2,\mathfrak{B}_{3,0}(x;a)=(\ln a)^3(1+x)^3,\mathfrak{B}_{3,1}(x;a)=(x+2x^2+x^3)(\ln a)^3$$

$$\mathfrak{B}_{3,2}(x;a)=(x^2+x^3)(\ln a)^3,\mathfrak{B}_{3,3}(x;a)=(x\ln a)^3$$

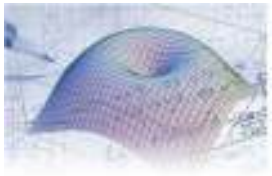
$$\mathfrak{B}_{n,0}(x;a)=(1+x)^n(\ln a)^n,\mathfrak{B}_{n,n-1}(x;a)=(x^{n-1}+x^n)(\ln a)^n,\mathfrak{B}_{n,n}(x;a)=x^n(\ln a)^n.$$

We also observe that

$$\mathfrak{B}_{n,k}(0;a)=\mathfrak{B}_{n,k}(-1;a)=0.$$

By (2.1), for $n\geq k$, we construct the generating function of the modified beta polynomials as follows

$$\mathfrak{b}_{k,a}(t,x):=\sum_{n=0}^{\infty}\mathfrak{B}_{n,k}(x;a)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(\ln a)^n\left(\frac{x}{1+x}\right)^k(1+x)^n\frac{t^n}{n!}=$$



$$=\left(\frac{x}{1+x}\right)^{k}\sum_{n=0}^{\infty}\left(\ln a\right)^{n}\left(1+x\right)^{n}\frac{t^{n}}{n!}=\left(\frac{x}{1+x}\right)^{k}e^{t(1+x)\ln a},$$

thus, we get

$$\mathfrak{b}_{k,a}\left(t,x\right)=\sum_{n=k}^{\infty}\mathfrak{B}_{n,k}\left(x;a\right)\frac{t^{n}}{n!}=\left(\frac{x}{1+x}\right)^{k}a^{t(1+x)}.$$

By (2.2), we see that

$$\mathfrak{b}_{k,a}\left(-t,x\right)\mathfrak{b}_{k,a}\left(t,x\right)=\left(\frac{x}{1+x}\right)^{2k}$$

and

$$\mathfrak{b}_{k,a}\left(t,x\right)a^{-t}=\left(\frac{x}{1+x}\right)^{k}a^{tx}.$$

For $k=0$ into (2.2) and $x\neq -1,$ we attain

$$\mathfrak{b}_{0,a}\left(t,x\right)=a^{t(1+x)},$$

which is the analytic functiıon for all $t\in\mathbb{C}.$ Also, for $k\neq 0,$ we get

$$\mathfrak{b}_{k,a}\left(t,0\right)=0$$

Using (2.2), we have

$$\mathfrak{b}_{k_1,a}\left(t,x\right)\mathfrak{b}_{k_2,a}\left(t,x\right)=\left(\frac{x}{1+x}\right)^{k_1}a^{t(1+x)}\left(\frac{x}{1+x}\right)^{k_2}a^{t(1+x)}$$

$$=\left(\frac{x}{1+x}\right)^{k_1+k_2}a^{2t(1+x)}=\mathfrak{b}_{k_1+k_2,a}\left(2t,x\right),$$

which gives the following theorem.

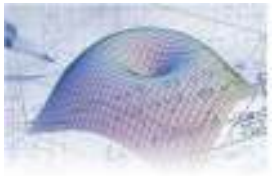
Theorem 1 *The following relationship*

$$\mathfrak{b}_{k_1,a}\left(t,x\right)\mathfrak{b}_{k_2,a}\left(t,x\right)=\mathfrak{b}_{k_1+k_2,a}\left(2t,x\right)$$

is valid for $n\in\mathbb{N}_0$.

By (2.4), we see that

$$\mathfrak{b}_{k_1,a}\left(-t,x\right)\mathfrak{b}_{k_2,a}\left(t,x\right)=\left(\frac{x}{1+x}\right)^{k_1+k_2}.$$



In view of (2.2), we observe that

$$\mathfrak{b}_{k,a}\left(t,x\right)=a^{t\left(1+x\right)}\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}\left(-1\right)^{m}x^{k+m},$$

where $\left|x\right|<1.$ Then we derive

$$\mathfrak{b}_{k,a}\left(t,-1\right)=\infty.$$

We observe that

$$\begin{aligned}\sum_{k=0}^{n}\left(-1\right)^{k}\mathfrak{B}_{n,k}\left(x;a\right)&=\sum_{k=0}^{n}\left(-1\right)^{k}\left(\ln a\right)^{n}\left(1+x\right)^{n-k}x^{k}\\&=\left(\ln a\right)^{n}\left(1+x\right)^{n}\sum_{k=0}^{n}\left(\frac{-x}{1+x}\right)^{k}\\&=\left(\ln a\right)^{n}\left(1+x\right)^{n}\frac{1-\left(\frac{-x}{1+x}\right)^{n+1}}{1+\frac{-x}{1+x}},\end{aligned}$$

which gives the following theorem.

Theorem 2 *For $n\geq k$, we acquire*

$$\sum_{k=0}^{n}\left(-1\right)^{k}\mathfrak{B}_{n,k}\left(x;a\right)=\left(\ln a\right)^{n}\left(1+x\right)^{n}\frac{\left(1+x\right)^{n+1}+\left(-1\right)^{n}x^{n+1}}{1+2x}.$$

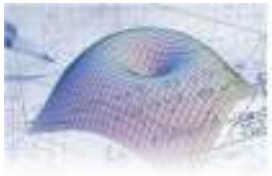
By (2.1) and for $\left|\frac{x}{1+x}\right|<1$, we get

$$\sum_{k=0}^{\infty}\mathfrak{B}_{n,k}\left(x;a\right)=\left(\ln a\right)^{n}\left(1+x\right)^{n}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{k}=\left(\ln a\right)^{n}\left(1+x\right)^{n+1},$$

which provides the following theorem.

Theorem 3 *The following summation formula*

$$\sum_{k=0}^{\infty}\mathfrak{B}_{n,k}\left(x;a\right)=\left(\ln a\right)^{n}\left(1+x\right)^{n+1}$$



holds for $\left|\frac{x}{1+x}\right|<1$.

Upon setting $k=0$ in (2.3) and utilizing (2.2), we attain

$$\sum_{n=k}^{\infty}(x\ln a)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{B}_{n,0}(x;a)\frac{t^n}{n!}\sum_{n=k}^{\infty}(-1)^n\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathfrak{B}_{n-l,0}(x;a)\frac{t^n}{n!},$$

which gives the following theorem.

Theorem 4 *The following formula including the modified beta polynomials*

$$x^n=\frac{1}{(\ln a)^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathfrak{B}_{n-l,0}(x;a)$$

is valid for $n\in\mathbb{N}_0$.

From (2.2), we attain

$$\sum_{n=k}^{\infty}\mathfrak{B}_{n,k}(x;a)\frac{t^n}{n!}=\left(\frac{x}{1+x}\right)^k\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(\ln a)^l x^l\frac{t^n}{n!}.$$

We arrive at the following theorem.

Theorem 5 *The following relation*

$$\mathfrak{B}_{n,k}(x;a)=(1+x)^{-k}\sum_{l=0}^{n}\binom{n}{l}(\ln a)^l x^{l+k}$$

holds for non-negative integers n.

By means of (2.2), we obtain

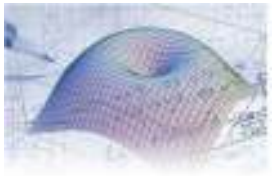
$$\sum_{k=0}^{\infty}\mathfrak{b}_{k,a}(t,x)=a^{t(1+x)}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k,$$

using the geometric series for $\left|\frac{x}{1+x}\right|<1$, then

$$\sum_{k=0}^{\infty}\mathfrak{b}_{k,a}(t,x)=a^{t(1+x)}\frac{1}{1-\frac{x}{1+x}}=(1+x)a^{t(1+x)},$$

and also

$$\frac{d}{dt}a^{t(1+x)}=\ln a(1+x)a^{t(1+x)},$$



which gives the following theorem.

Theorem 6 *Let* $\left|\frac{x}{1+x}\right|<1$. *Then, we have*

$$\frac{d}{dt}a^{t(1+x)}=\frac{1}{\ln a}\sum_{k=0}^{\infty}\mathfrak{b}_{k,a}\left(t,x\right)$$

Now, we give some partial differential equations for the modified beta polynomials as follows.

Taking the derivative of (2.2) with respect to x, we acquire the following PDEs:

$$\frac{\partial\mathfrak{b}_{k,a}\left(t,x\right)}{\partial x}=\frac{k}{x^2}\mathfrak{b}_{k+1,a}\left(t,x\right)+t\ln a\mathfrak{b}_{k,a}\left(t,x\right)$$

and

$$\frac{\partial\mathfrak{b}_{k,a}\left(t,x\right)}{\partial x}=\frac{k}{\left(1+x\right)^2}\mathfrak{b}_{k-1,a}\left(t,x\right)+t\ln a\mathfrak{b}_{k,a}\left(t,x\right)$$

By means of (2.2), we attain

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{d}{dx}\mathfrak{B}_{n,k}\left(x;a\right)\frac{t^n}{n!}&=\frac{d}{dx}\left(\frac{x}{1+x}\right)^k a^{t(1+x)}\\&=a^{t(1+x)}\frac{k}{x^2}\left(\frac{x}{1+x}\right)^{k+1}+\left(\frac{x}{1+x}\right)^k a^{t(1+x)}t\ln a\\&=\frac{k}{x^2}\sum_{n=k}^{\infty}\mathfrak{B}_{n,k+1}\left(x;a\right)\frac{t^n}{n!}+\ln a\sum_{n=k}^{\infty}\mathfrak{B}_{n,k}\left(x;a\right)\frac{t^{n+1}}{n!},\end{aligned}$$

which gives the following theorem.

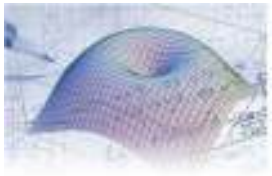
Theorem 7 *The following derivative formula*

$$\frac{d}{dx}\mathfrak{B}_{n,k}\left(x;a\right)=\frac{k}{x^2}\mathfrak{B}_{n,k+1}\left(x;a\right)+n\mathfrak{B}_{n-1,k}\left(x;a\right)$$

holds for $n\geq 1$.

From (2.2), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{d}{dx}\mathfrak{B}_{n,k}\left(x;a\right)\frac{t^n}{n!}&=\frac{d}{dx}\left(\frac{x}{1+x}\right)^k a^{t(1+x)}\\&=a^{t(1+x)}\frac{k}{\left(1+x\right)^2}\left(\frac{x}{1+x}\right)^k+\left(\frac{x}{1+x}\right)^k a^{t(1+x)}t\ln a\end{aligned}$$



$$= \frac{k}{(1+x)^2} \sum_{n=k}^{\infty} \mathfrak{B}_{n,k-1}(x;a) \frac{t^n}{n!} + \ln a \sum_{n=k}^{\infty} \mathfrak{B}_{n,k}(x;a) \frac{t^{n+1}}{n!},$$

which gives the following theorem.

Theorem 8 *The following derivative formula*

$$\frac{d}{dx} \mathfrak{B}_{n,k}(x;a) = \frac{k}{(1+x)^2} \mathfrak{B}_{n,k-1}(x;a) + n \mathfrak{B}_{n-1,k}(x;a)$$

holds for $n \geq 1$.

Conclusion

In the present study, we have introduced modified beta-type polynomials. We have then constructed generating function for modified beta polynomials. By utilizing this generating function, we have investigated some basic properties of these polynomials. Moreover, we have provided several fundamental equations and partial differential equations (PDEs) associated with this generating function.

After this study, one can resarch this polynomials under the operatinal methods or umbral calculus to get more formulas and properties. And also one can investigate corellations for modified beta-type polynomials in conjuction with the Bernoulli, Euler, Hermite polynomials and so on.

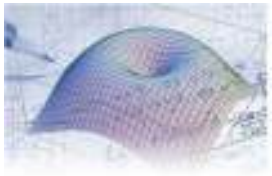
References

[1] Y. Simsek, “Beta-type polynomials and their generating functions” Appl. Math. Comput. 254(2015), 172-182.

[2] Y. Simsek, “Combinatorial sums and binomial identities associated with the Beta-type polynomials” Hacet. J. Math. Stat. 47(2018), 1144-1155.

[3] Y. Simsek, “Matrix Representation for the Beta Type Polynomials” Appl. Math. Inf. Sci. 11(2017), 183-187.

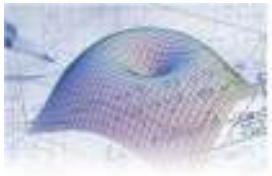
[4] M. Chand, H. Hachimi, R. Rani, “New Extension of Beta Function and Its Applications”, Int. J. Math. Math. Scie. V. 2018, Article ID 6451592, 25 pages.



[5] U. Duran, "Modified Sumudu Transform and Its Properties", Sakarya Univ. J. Sci., 25(2021), 389-396.

[6] M. Saif, F. Khan, K. S. Nisar, S. Araci, ”Modified Laplace transform and its properties,” J. Math. Comput. Sci., 21(2020), 127-135.

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



On β-Hermite-Bell Polynomials

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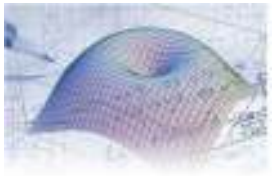
Abstract: In this study, we introduce β-extension of Hermite-Bell polyomials and then, investigate some of their porperties and identities. Also, we derive several relations with some other well-known special polynomials. Moreover, we provide a derivative property and an integral representation for these polynomials.

Keywords: Hermite polynomials, Bell polynomials, Hermite-Bell polynomials, modified exponential function, generating functions

Introduction

Throughout this paper, the familiar symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers, and the set of all non-negative integers, respectively.
The Bernoulli polynomials $B_n(x)$, the Euler polynomials $E_n(x)$ and the Genocchi polynomials $G_n(x)$ are defined as follows (*cf.* [7]):

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt} \quad (|t|<2\pi),$$



$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt} \quad (|t|<\pi).$$

and

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt} \quad (|t|<\pi).$$

Upon setting $x=0$ in (1.1), (1.2) and (1.3), we get $B_n(0) := B_n$, $E_n(0) := E_n$ and $G_n(0) := G_n$ known as the usual Bernoulli numbers, the Euler numbers and the Genocchi numbers, respectively.

The Stirling polynomials $S_2(n,k:x)$ and numbers $S_2(n,k)$ of the second kind are given by the following exponential generating functions (*cf.* [1-4,7-11]):

$$\sum_{n=0}^{\infty} S_2(n,k:x)\frac{t^n}{n!} = \frac{\left(e^t-1\right)^k}{k!}e^{tx} \quad \text{and} \quad \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{\left(e^t-1\right)^k}{k!}.$$

The Stirling numbers of the second kind can also be derived by the following recurrence relation for $n \in \mathbb{N}_0$ (*cf.* [1-4,7-11]):

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$ for $n \in \mathbb{N}$ with $(x)_0 = 1$.

The bivariate Bell polynomials are defined as follows:

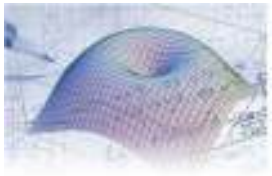
$$\sum_{n=0}^{\infty} Bel_n(x;y)\frac{t^n}{n!} = e^{y\left(e^t-1\right)}e^{xt}$$

When $x=0$, $Bel_n(0;y) := Bel_n(y)$ called the classical Bell polynomials given by means of the following generating function (*cf.* [1-4,8-11]):

$$\sum_{n=0}^{\infty} Bel_n(y)\frac{t^n}{n!} = e^{y\left(e^t-1\right)}.$$

The Bell numbers Bel_n are acquired by taking $y=1$ in (1.7), that is $Bel_n(0;1) = Bel_n(1) := Bel_n$ and are given by the following exponential generating function (*cf.* [9]):

$$\sum_{n=0}^{\infty} Bel_n\frac{t^n}{n!} = e^{\left(e^t-1\right)}.$$



The Bell polynomials have been intensely investigated and studied by several mathematicians, *cf*. [1-4,8-11] and see also the references cited therein.
The Hermite polynomials are defined by (*cf*. [1-4]):

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{2xt-t^2}.$$

In this study, we introduce β-extension of Hermite-Bell polynomials and then, investigate some of their porperties and identities. Moreover, we provide two derivative property for these polynomials. Also, we examine some relationships with some well known special polynomials such as Bernoulli, Genocchi and Euler polynomials.

β-Hermite-Bell Polynomials

Recently, modified (β-) version of some integral transforms are studied and some of their properties and applications are invesitgated in [5,6].
In [6], the β-Laplace transform of a function $f(t)$ which is peace-wise continuous and of exponential order is considered as follows

$$_{\beta}\left(f(t)\right) = F(s;\beta) = \int_0^{\infty} \beta^{-st} f(t)\,dt,$$

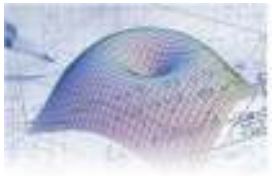
where $Re(s) > 0$ and $\beta \in (0,\infty)\setminus\{1\}$. Note that upon setting $\beta = e$, the β-Laplace transform reduces to usual Laplace transform. Then the authors gave several basic properties of β-Laplace transform and provided connections with different functions in [6].
Motivated by the above, the β-Sumudu transform is defined as follows.

Definition 1 *Let* $\beta \in (0,\infty)\setminus\{1\}$ *and*

$$A_{\beta} = \left\{ \begin{array}{c} f(t) \,\middle|\, \exists M, \tau_1, \tau_2 > 0 \text{ such that} \\ |f(t)| < M\beta^{1/\tau_j}(|t|), \text{if } t \in (-1)^j \times [0,\infty) \end{array} \right\}.$$

Then, for $f(t) \in A_a$, the β-Sumudu transform is considered by the following improper integral:

$$\mathcal{G}_{\beta}(u) = \mathcal{S}_{\beta}\left[f(t)\right] = \frac{1}{u}\int_0^{\infty} \beta^{\frac{-s}{u}} f(t)\,dt, u \in (-\tau_1, \tau_2).$$



Note that

$$\mathcal{S}_e\left[f(t)\right] := \mathbf{S}\left[f(t)\right].$$

By the same motivation, we now introduce the β-extension of Hermite-Bell polynomials follows.

Definition 2 *Let $\beta \in (0,\infty)\setminus\{1\}$. The β-Hermite-Bell polynomials are introduced by the following exponential generating function:*

$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!} = \beta^{\left(x^r-(x-t)^r\right)}.$$

Remark 1 *In the special case $\beta = e$ in (2.4), we acquire the usual Hermite-Bell polynomials as follows (cf. [1-4]):*

$$\sum_{n=0}^{\infty} H_{n}^{r}(x)\frac{t^n}{n!} = e^{\left(x^r-(x-t)^r\right)}.$$

Remark 2 *In the special case $r = 1$ in (2.4), we attain the generalized exponential function as follows (cf. [1-4]):*

$$\sum_{n=0}^{\infty} H_{n,\beta}^{1}(x)\frac{t^n}{n!} = \beta^{t}.$$

Remark 3 *In the special case $r = 2$ in (2.4), we obtain the generalized exponential function as follows (cf. [1-4]):*

$$\sum_{n=0}^{\infty} H_{n,\beta}^{2}(x)\frac{t^n}{n!} = \beta^{2xt-t^2},$$

which is the β-extension of the Hermite polynomials in (1.9).

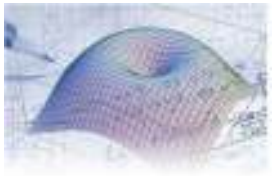
Upon letting $x = 0$ in (2.4), the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ satisfy the following relation

$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(0)\frac{t^n}{n!} = \beta^{-(-t)^r} = e^{-(-t)^r \ln\beta} = \sum_{n=0}^{\infty}\left(\ln\beta^{-1}\right)^n(-1)^{rn}\frac{t^{rn}}{n!},$$

which gives the following relation:

$$H_{nr,\beta}^{r}(0) = \frac{(nr)!}{n!}\left(\ln\beta^{-1}\right)^n(-1)^{rn}.$$

Setting $r = 1$ in (2.4), the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ satisfy the following relation



$$\sum_{n=0}^{\infty} H_{n,\beta}^{1}(x)\frac{t^n}{n!} = \beta^{(x-(x-t))} = \beta^{t} = \sum_{n=0}^{\infty} (\ln\beta)^n \frac{t^n}{n!},$$

which provides the following relation:

$$H_{n,\beta}^{1}(x) = (\ln\beta)^n \quad (n \in \mathbb{N}_0).$$

Also, by (2.4), we observe that

$$H_{0,\beta}^{r}(x) = 1,$$

which shows that the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ is an Appell polynomials. So, some of properties of the polynomials $H_{n,\beta}^{r}(x)$ can be investigated under the umbral calculus and operational methods. But, here we invetigate some properties of these polynomials by means of the series manipulation methods by using their generating function in (2.4).

For $\omega \in \mathbb{C}$ with $\omega^r = 1$, by (2.4), we have

$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!} = \beta^{\left(x^r-(x-t)^r\right)} = \beta^{\left((\omega x)^r-(\omega x-\omega t)^r\right)} = \sum_{n=0}^{\infty} H_{n,\beta}^{r}(\omega x)\frac{\omega^n t^n}{n!},$$

which gives the following theorem.

Theorem 1 *For $\omega \in \mathbb{C}$ with $\omega^r = 1$, we obtain*

$$H_{n,\beta}^{r}(x) = \omega^n H_{n,\beta}^{r}(\omega x).$$

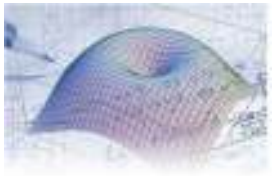
By (2.4), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{d}{dx} H_{n,\beta}^{r}(x)\frac{t^n}{n!} &= \frac{d}{dx}\beta^{\left(x^r-(x-t)^r\right)} \\
&= \beta^{\left(x^r-(x-t)^r\right)} \ln\beta \frac{d}{dx}\left(x^r-(x-t)^r\right) \\
&= \beta^{\left(x^r-(x-t)^r\right)} \ln\beta \left(rx^{r-1}-r(x-t)^{r-1}\right) \\
&= r\ln\beta \sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!}\left(x^{r-1}-(x-t)^{r-1}\right),
\end{aligned}$$

which provides the following derivative formula.

Theorem 2 *The following derivative formula*

$$\frac{d}{dx} H_{n,\beta}^{r}(x) = \ln\beta^r \left(x^{r-1}-(x-t)^{r-1}\right) H_{n,\beta}^{r}(x)$$



is valid for $n \in \mathbb{N}_0$.

By (2.4), we have

$$\sum_{n=0}^{\infty} \frac{d}{dt} H_{n,\beta}^{r}(x) \frac{t^n}{n!} = \frac{d}{dt} \beta^{\left(x^r-(x-t)^r\right)}$$

$$= \beta^{\left(x^r-(x-t)^r\right)} \ln \beta \frac{d}{dt}\left(x^r-(x-t)^r\right)$$

$$= \beta^{\left(x^r-(x-t)^r\right)} \ln \beta \left(r(x-t)^{r-1}\right)$$

$$= r \ln \beta (x-t)^{r-1} \sum_{n=0}^{\infty} H_{n,\beta}^{r}(x) \frac{t^n}{n!},$$

which gives the following theorem.

Theorem 3 *The following derivative formula*

$$\frac{d}{dt} H_{n,\beta}^{r}(x) = \ln \beta^r (x-t)^{r-1} H_{n,\beta}^{r}(x)$$

is valid for $n \in \mathbb{N}_0$.

We now perform to acquire several representations for $H_{n,\beta}^{r}(x)$ by means of the familiar Bernoulli, Euler and Genocchi numbers and polynomials.

By (1.1) and (2.4), we see that

$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x) \frac{t^n}{n!} = \beta^{\left(x^r-(x-t)^r\right)} \frac{t}{e^t-1} \frac{e^t-1}{t}$$

$$= \sum_{n=0}^{\infty} H_{n,\beta}^{r}(x) \frac{t^n}{n!} \left(\sum_{n=0}^{\infty} B_n(1) \frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty} B_n \frac{t^{n-1}}{n!} \right)$$

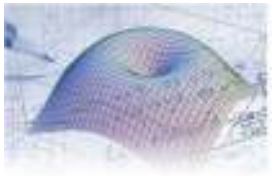
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} H_{n-k,\beta}^{r}(x) \left(B_k(1) - B_k\right) \frac{t^{n-1}}{n!},$$

which provides the following relationship including the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ and usual Bernoulli polynomials and numbers.

Theorem 4 *For* $n \in \mathbb{N}$, *we have*

$$H_{n,\beta}^{r}(x) = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} H_{n+1-k,\beta}^{r}(x) \left(B_k(1) - B_k\right).$$

By (1.2) and (2.4), we observe that



$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!} = \beta^{\left(x^r-(x-t)^r\right)}\frac{2}{e^t+1}\frac{e^t+1}{2}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty} E_n(1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} H_{n-k,\beta}^{r}(x)\left(E_k(1)+E_k\right)\frac{t^{n-1}}{n!},$$

which gives the following relation including the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ and familiar Euler polynomials and numbers.

Theorem 5 *For $n \in \mathbb{N}$, we have*

$$H_{n,\beta}^{r}(x) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} H_{n-k,\beta}^{r}(x)\left(E_k(1)+E_k\right).$$

By (1.1) and (2.4), we see that

$$\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!} = \beta^{\left(x^r-(x-t)^r\right)}\frac{2t}{e^t+1}\frac{e^t+1}{2t}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} H_{n,\beta}^{r}(x)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty} G_n(1)\frac{t^{n-1}}{n!} + \sum_{n=0}^{\infty} G_n\frac{t^{n-1}}{n!}\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} H_{n-k,\beta}^{r}(x)\left(G_k(1)+G_k\right)\frac{t^{n-1}}{n!},$$

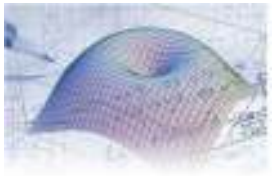
which provides the following correlation covering the β-Hermite-Bell polynomials $H_{n,\beta}^{r}(x)$ and usual Genocchi polynomials and numbers.

Theorem 6 *For $n \in \mathbb{N}$, we have*

$$H_{n,\beta}^{r}(x) = \frac{1}{2(n+1)}\sum_{k=0}^{n+1}\binom{n+1}{k} H_{n+1-k,\beta}^{r}(x)\left(G_k(1)+G_k\right).$$

Conclusion

In the present paper, we have introduced β-extension of Hermite-Bell polyomials and then, investigated some of their porperties and identities. Moreover, we have provided two derivative property for these polynomials. Also, we have examined some



relationships with some well known special polynomials such as Bernoulli, Genocchi and Euler polynomials.

References

[1] E. Hetmaniok, M. Pleszczyński, D. Slota, R. W., "On similarities between exponential polynomials and Hermite polynomials" J. Appl. Math. Comput. Mechanics, 12(2013), 93-104.

[2] M. W. Kalinowski, M. Seweryński, "Differential equation for Hermite-Bell polynomials", Math. Proc. Phil. Soc. 91(1982), 259-265.

[3] D. Dominici, "Asymptotic analysis of generalized Hermite polynomials", Analysis 28(2008), 239-261.

[4] R. B. Paris, "The asymptotics of the generalized Hermite-Bell polynomials", J. Comp. Appl. Math., 232(2009), 216-226.

[5] U. Duran, "Modified Sumudu Transform and Its Properties", Sakarya Univ. J. Sci., 25(2021), 389-396.

[6] M. Saif, F. Khan, K. S. Nisar, S. Araci, "Modified Laplace transform and its properties," J. Math. Comput. Sci., 21(2020), 127-135.

[7] H. M. Srivastava, A. Pinter, "Remarks on some relationships between the Bernoulli and Euler polynomials," Appl. Math.Lett. 17 (2004), 375-380.

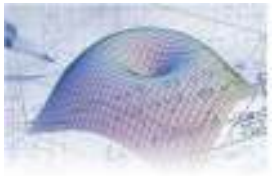
[8] E.T. Bell, Exponential polynomials. Ann. Math. 35 (1934) 258-277.

[9] L. Carlitz, Some remarks on the Bell numbers. Fibonacci Quart. 18 (1980) 66-73.

[10] D.S. Kim, T. Kim, Some identities of Bell polynomials, Sci. China Math. 58 (2015) 2095-2104.

[11] T. Kim, D. S. Kim, L.-C. Jang, H. Lee, H.-Y. Kim, Complete and incomplete Bell polynomials associated with Lah-Bell numbers and polynomials. Adv Differ Equ 2021, 101 (2021).

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



Berezin number and strongly convex functions

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Abstract: In this work, using the operator extension of the Jensen inequality and Hölder-McCarthy inequality, we obtain Berezin number inequalities for f(A), where A is a self-adjoint operaor on the reproducing kernel Hilbert space and f is a strongly convex function.

Keywords: Berezin number, Berezin symbol, self-adjoint operator, strongly convex function.

1. Introduction

Let $J \subset \mathbb{R}$ be an interval and let c be a positive number. A function $f : J \to \mathbb{R}$ is said to be strongly convex with modulus c if

$$f(\eta a+(1-\eta)b) \le \eta f(a)+(1-\eta) f(b)-c\eta(1-\eta)(a-b)^2$$

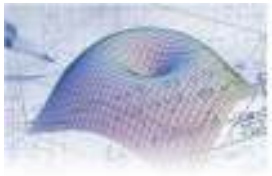
for every $a,b \in J$ and $\eta \in [0,1]$ (see [14]). A function $f : J \to \mathbb{R}$ is strongly convex with modulus c if and only if for every $a_0 \in \operatorname{int}(J)$ (the interior of J) there exists a number $l \in \mathbb{R}$ such that

$$(1)\ c(a-a_0)^2 + l(a-a_0) + f(a_0) \le f(a), a \in J.$$

The differentiable function f is strongly convex with modulus c if and only if

$$(2)\ \left(f'(a)-f'(b)\right)(a-b) \ge 2c(a-b)^2$$

for all $a,b \in J$.



Let A be a selfadjoint linear operator on a complex Hilbert space H. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathsf{S}_p(A))$ of all continuous functions defined on the spectrum of A, denoted by $\mathsf{S}_p(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [6]:

For any $f, g \in C(\mathsf{S}_p(A))$ and any $\alpha, \beta \in \mathsf{C}$ we have

$$\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g);\ (i)$$

$$\Phi(fg) = \Phi(f)\Phi(g) \text{ and } \Phi(\overline{f}) = \Phi(f)^*;\ (ii)$$

$$\|\Phi(f)\| = \|f\| = \sup_{t\in\mathsf{S}_p(A)} |f(t)|;\ (iii)$$

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathsf{S}_p(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(\mathsf{S}_p(A))$ and we call it the continuous functional calculus for a selfadjoint operator A. If A is a selfadjoint operator and f is a real-valued continuous function on $\mathsf{S}_p(A)$, then $f(t) \geq 0$ for any $t \in \mathsf{S}_p(A)$ implies that $f(A) \geq 0$. Moreover, if both f and g are real-valued functions on $\mathsf{S}_p(A)$ then the following important property holds:

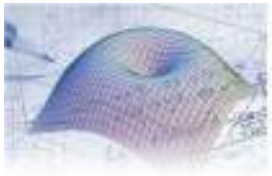
$$f(t) \geq g(t) \text{ for any } t \in \mathsf{S}_p(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $\mathbf{B}(H)$.

The following result that provides an operator version for Jensen's inequality is due to Mond and Pecaric [13].

Lemma 1. *Let $f(t)$ be a real valued continuous convex function, and let A be a self-adjoint operator on a Hilbert space H. Then Jensen's inequality asserts that*

$$f\langle Ax, x\rangle \leq \langle f(A)x, x\rangle$$



for any unit vector $x \in H$.

Lemma 2. *Let* $A \in \mathbf{B}(\mathbf{H})$, $A \geq 0$ *and let* $x \in \mathbf{H}$ *be any unit vector. Then*

$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ *for* $r \geq 1$; (a)

$\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ *for* $0 < r \leq 1$. (b)

Recall that a reproducing kernel Hilbert space (shorty, RKHS) is the Hilbert space $\mathbf{H} = \mathbf{H}(\Omega)$ of complex-valued functions on some set Ω such that:

(a) the evaluation functional $f \to f(\lambda)$ is continuos for each $\lambda \in \Omega$;

(b) for any $\lambda \in \Omega$ there exists $f_\lambda \in \mathbf{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then by the classical Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathbf{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathbf{H}$. The function k_λ is called the reproducing kernel of the space $\mathbf{H}$. It is well known that (see [2,15])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathbf{H}(\Omega)$. Let

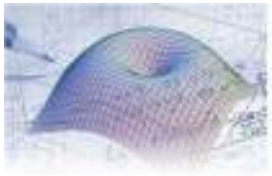
$$\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$$

denote the normalized reproducing kernel of the space $\mathbf{H}$ (note that by (b), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the RKHS $\mathbf{H}$, its Berezin symbol $\widetilde{A}$ is defined by the formula (see [3])

$$\widehat{A}(\lambda) := \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle_{\mathbf{H}} \ (\lambda \in \Omega).$$

The Berezin symbol is a function that is bounded by the norm of the operator.

Berezin set and Berezin number of operator A are defined by respectively (see Karaev [9,10])



$$Ber(A) := Range(\widehat{A}) = \{\widehat{A}(\lambda) : \lambda \in \Omega\} \text{ and } ber(A) := \sup\{|\widehat{A}(\lambda)| : \lambda \in \Omega\}.$$

Recall that $W(A) := \{\langle Af, f\rangle : \|f\|_{\mathbf{H}} = 1\}$ is the numerical range of the operator A and

$$w(A) := \sup\{|\langle Af, f\rangle| : \|f\|_{\mathbf{H}} = 1\}$$

is the numerical radius of A (see for detail information, [1, 5, 11, 12, 23]). It is well known that

$$ber(A) \leq w(A) \leq \|A\|$$

and

$$\frac{\|A\|}{2} \leq w(A)$$

for any $A \in \mathbf{B}(\mathbf{H})$. But, Karaev in [10] showed that

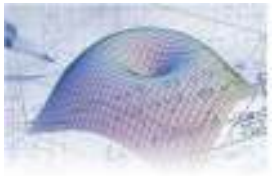
$$\frac{\|A\|}{2} \leq ber(A)$$

is not hold for every $A \in \mathbf{B}(\mathbf{H})$. Some important result related with Berezin number inequality were obtained by authors in [4, 16, 17, 20, 22]. Also, Berezin number inequalities were given by using Hardy-Hilbert type inequalities in [7, 8, 18, 19, 21].

In this paper, using the operator extension of the Jensen inequality and Hö lder-McCarthy inequality, we obtain Berezin number inequalities for $f(A)$, where A is a self-adjoint operaor on the reproducing kernel Hilbert space and f is a strongly convex function.

2. Results

The main result of this section is the following:



Theorem 1. *Let* $\mathbf{H} = \mathbf{H}(\Omega)$ *be a RKHS,* $A \in \mathbf{B}(\mathbf{H})$ *be a selfadjoint operator with* $\mathsf{S}(A) \subseteq \mathrm{int}(J)$ *and* $f : J \to \mathsf{R}$ *be strongly convex with modulus* c *and differentiable on* $\mathrm{int}(J)$. *Then*

$$f\left(ber\left(A\right)\right) + cber\left(A^2\right) \le ber\left(f\left(A\right)\right) + cber^2\left(A\right).$$

Proof By using functional calculus for (2), we have

$$c\left(A + a_0^2 I - 2a_0 T\right) + lT + la_0 I + f\left(a_0\right) I \le f\left(T\right)$$

which is equivalent to

$$c\left(\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + a_0^2 - 2a_0\right) + l\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - la_0 + f\left(a_0\right) \le \left\langle f\left(A\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

for all $\lambda \in \Omega$.

Putting $a_0 = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ above inequality, we get

$$f\left(\widehat{A}\left(\lambda\right)\right) \le \widehat{f\left(A\right)}\left(\lambda\right) - c\left(\widehat{A^2}\left(\lambda\right) - \widehat{A}\left(\lambda\right)^2\right)$$

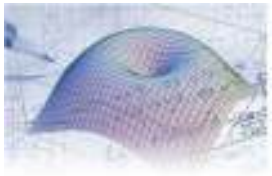
and hence

$$(3)\ f\left(\widehat{A}\left(\lambda\right)\right) + c\widehat{A^2}\left(\lambda\right) \le f\left(\widehat{A}\left(\lambda\right)\right) + c\widehat{A}\left(\lambda\right)^2$$

for all $\lambda \in \Omega$. Taking supremum on $\lambda \in \Omega$ above inequality, we have

$$f\left(ber\left(A\right)\right) + cber\left(A^2\right) \le ber\left(f\left(A\right)\right) + cber^2\left(A\right).$$

Theorem 2. $f : J \to \mathsf{R}$ *be strongly convex with modulus* c. *If* f^v *with* $v \in [0,1]$ *is strongly convex with modulus* c, *then*



$$\begin{aligned} f\left(\widehat{A}(\lambda)\right) &\le f^{1-v}\left(\widehat{A}(\lambda)\right) f^{v}\left(\widehat{A}(\lambda)\right) + c^{'} f^{1-v}\left(\widehat{A}(\lambda)\right)\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right) \\ &\le f^{1-v}\left(\widehat{A}(\lambda)\right)\widehat{f^{v}(A)}(\lambda) \\ &\le f^{1-v}\left(\widehat{A}(\lambda)\right)\left[\widehat{f(A)}(\lambda)\right]^{v} \\ &\le (1-v) f\left(\widehat{A}(\lambda)\right) + v\widehat{f(A)}(\lambda) \\ &\le \widehat{f(A)}(\lambda) - c(1-v)\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right) \\ &\le \widehat{f(A)}(\lambda) \end{aligned}$$

for any positive operator A *with* $\mathsf{S}(A) \subseteq J$ *and* $\lambda \in \Omega$.

Proof. Since f^{v} is strongly convex, we get from (3)

$$f^{v}\left(\widehat{A}(\lambda)\right) \le \widehat{f^{v}(A)}(\lambda) - c\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right)$$

for all $\lambda \in \Omega$. Multiplying both sides by $f^{1-v}\left(\widehat{A}(\lambda)\right)$, we have that

$$\begin{aligned} f\left(\widehat{A}(\lambda)\right) &= f^{1-v}\left(\widehat{A}(\lambda)\right) f^{v}\left(\widehat{A}(\lambda)\right) \\ &\le f^{1-v}\left(\widehat{A}(\lambda)\right)\widehat{f^{v}(A)}(\lambda) - cf^{1-v}\left(\widehat{A}(\lambda)\right)\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right) \end{aligned}$$

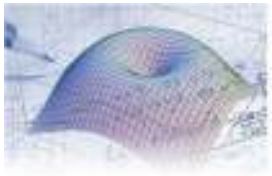
for all $\lambda \in \Omega$. Rearranging the terms, we get that

$$\begin{aligned} f\left(\widehat{A}(\lambda)\right) &\le f^{1-v}\left(\widehat{A}(\lambda)\right) f^{v}\left(\widehat{A}(\lambda)\right) + cf^{1-v}\left(\widehat{A}(\lambda)\right)\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right) \\ &\le f^{1-v}\left(\widehat{A}(\lambda)\right)\widehat{f^{v}(A)}(\lambda) \end{aligned}$$

for all $\lambda \in \Omega$. On the other hand, by using the Hölder-McCarthy inequality, we obtain

$\widehat{f^{v}(A)}(\lambda) \le \left[f\left(\widehat{A}(\lambda)\right)\right]^{v}$.

Multiplying both sides by $f^{1-v}\left(\widehat{A}(\lambda)\right)$, we have



$$f^{1-v}\left(\widehat{A}(\lambda)\right)\widehat{f^{v}(A)}(\lambda) \le f^{1-v}\left(\widehat{A}(\lambda)\right)\left[f\left(\widehat{A}(\lambda)\right)\right]^{v}$$
$$\le (1-v) f\left(\widehat{A}(\lambda)\right) + vf\left(\widehat{A}(\lambda)\right)$$
$$\le f\left(\widehat{A}(\lambda)\right) - c(1-v)\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right)$$

Combining above inequalities, we have the desired result.

Theorem 3. *Let A be a self-adjoint operator on reproducing kernel Hilbert spaces with $\sigma(A) \subset \mathrm{int}(J)$. If $f : J \to \mathsf{R}$ is strongly convex with modulus c and differentiable on $\mathrm{int}(J)$, and the derivative of f is continuous on $\mathrm{int}(J)$, then*

$$\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2 \le \frac{1}{2c}\left(\widehat{f'(A)A}(\lambda) - \widehat{A}(\lambda)\widehat{f'(A)}(\lambda)\right)$$

for all $\lambda \in \Omega$.

Proof. Utilizing functional calculus for (2), we get

$$2c\left(A^2 - y^2 I + 2A^2 y\right) \le f'(A)A - yf'(A) - f'(y)A + yf'(y)I$$

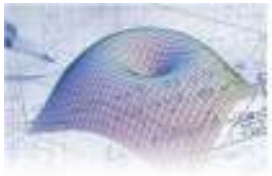
which is equivalent to

$$2c\left(\widehat{A^2}(\lambda) + y^2 - 2y\widehat{A}(\lambda)\right) \le \widehat{f'(A)A}(\lambda) - y\widehat{f'(A)}(\lambda) - f'(y)\widehat{A}(\lambda) + yf'(y)$$

for all $\lambda \in \Omega$. Putting the $y = \widehat{A}(\lambda)$, we obtain

$$2c\left(\widehat{A^2}(\lambda) - \widehat{A}(\lambda)^2\right) \le \widehat{f'(A)A}(\lambda) - \widehat{A}(\lambda)\widehat{f'(A)}(\lambda),$$

which gives the desired result.



REFERENCES

[1] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for products and commutators of operators, Houston J. Math., 41(4)(2015), 1163-1173.

[2] N. Aronzajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950), 337 - 404.

[3] F.A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv., 6 (1972), 1117-1151.

[4] M. Bakherad, Some Berezin Number Inequalities for Operator Matrices, Czech. Math. J., 68(2018), 997-1009.

[5] S.S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces, Banach J. Math. Anal., 1(2)(2007), 154-175.

[6] T. Furuta, J. Mićić Hot, J. Pečarić, Y. Seo, Mond-Pečarić method in operator inequalities (Inequalities for bounded selfadjoint operators on a Hilbert space), Element, Zagreb, 2005.

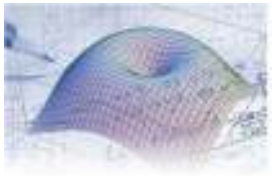
[7] M. T. Garayev, M. Gürdal, S. Saltan, Hardy type inequaltiy for reproducing kernel Hilbert space operators and Related Problems, Positivity, 21(4)(2017), 1615-1623.

[8] M. T. Garayev, M. Gürdal, A. Okudan, Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators, Math. Inequal. Appl., 19(3)(2016), 883-891.

[9] M.T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, J. Funct. Anal., 238(2006), 181-192.

[10] M.T. Karaev, Reproducing kernels and Berezin symbols techniques in various questions of operator theory, Complex Anal. Oper. Theory, 7(2013), 983-1018.

[11] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci., 24(2)(1988), 283-293.



[12] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math., 168(1)(2005), 73-80.

[13] B. Mond, J. Pecaric, Convex inequalities in Hilbert space, Houston J. Math., 19(1993), 405-420.

[14] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math. Dokl., 7 (1966), 72-75.

[15] S. Saitoh, Y. Sawano, Theory of reproducing kernels and applications, Springer, 2016.

[16] S. Sahoo, N. Das, D. Mishra, Berezin number and numerical radius inequalities for operators on Hilbert spaces, Adv. Oper. Theory, 5(2020), 714-727.

[17] Taghavi A. , Roushan T.A., Darvish V. Some upper bounds for the Berezin number of Hilbert space operators. Filomat. 33(2019), 4353-4360.

[18] U. Yamancı, M. Gürdal, On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space, New York J. Math, 23(2017), 1531-1537.

[19] U. Yamancı, M. Gürdal, M. T. Garayev, Berezin number inequality for convex function in reproducing kernel Hilbert space, Filomat, 31(18)(2017), 5711-5717.

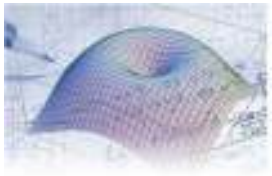
[20] U. Yamancı, M. Garayev, Some results related to the Berezin number inequalities, Turk. J. Math., 43(2019), 1940-1952.

[21] U. Yamancı, M. Garayev, C. Çelik, Hardy-Hilbert type inequality in reproducing kernel Hilbert space: its applications and related results, Linear Multilinear Algebra, 67(2019), 830-842.

[22] U. Yamancı, R. Tunç , M. Gürdal, Berezin number, Grü ss-type inequalities and their applications, Bull. Malays. Math. Sci. Soc., 43(2020), 2287-2296.

[23] T. Yamazaki, On upper and lower bounds of the numerical radius and equality condition, Studia Math., 178(2007), 83-89.

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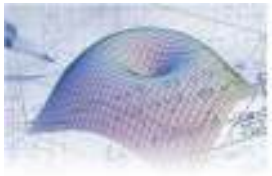


IMPLEMENTING BOOTSTRAP AND MULTIPLE LOGISTIC REGRESSION IN CARDIOVASCULAR RESEARCH

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Abstract
Sometimes it is difficult to implement advanced statistics in research areas such as medicine especially for a non statistical expert. Nowadays bootstrapping is becoming a useful and very popular tool in wide areas of research, and also in other statistical applications such as regression. An interesting use of nonparametric bootstrap is in regression models. In our study we will use medical records in cardiology and will perform a stepwise multiple logistic regression is to identify risk factors and we will also implement bootstrap technique with intention to develop a better use and implementation of these techniques in practice.
Key words: bootstrap, logistic regression, estimation, cardiology, data records

INTRODUCTION
Nowadays it is very difficult to apply advanced statistics in research areas such as medicine especially for a non statistical expert. We are focused in implementing advanced statistics elements in a study conducted with data records from a hospital with intention not only to identify the significant risk factors in mortality of the patients, but also to use bootstrap in obtaining the estimation for the model coefficients.
Bootstrap is based on resampling of the data and since its introduction by Efron (1979), bootstrap has resulted a useful tool in estimating parameters and constructing confidence intervals for the parameters.
Considering 15 covariates, we constructed a logistic regression model in the case of studying the mortality of the patients hospitalized. We used both R programming language and SPSS to obtain the logistic regression model and



also we used nonparametric bootstrap to obtain the coefficient estimations for the model.

There are some difficulties that associate our study. First, it is not easy to find an appropriate logistic regression model to determine the significant factors and to obtain important conclusions in the area of study.

Second, implementing both: logistic regression and bootstrap it is considered a difficult task, but in our study we obtained good results and a premise to continue applying bootstrap in similar studies.

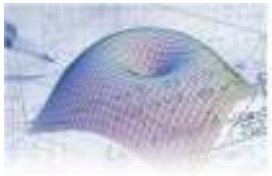
OBJECTIVES

The aim of this paper is to implement bootstrap methods and logistic regression in a study with data from cardiology. From the medical point of view, we want to obtain the significant factors in the model and to identify the risk factors for mortality of the patients. From a statistical point of view we want to examine the impact of using bootstrap methods in estimating the coefficients of the model and compare the results using both R and SPSS.

METHODOLOGY

1. Data in the case study

We used the data records from the Cardiology Department in a hospital in Tirana, Albania, with a total of 2201 patients that include the following variables: Gender, Age, Coronary Artery Disease, Diabetes Mellitus, Ejection Fraction, Body Mass Index, Smoking, Dyslipidemia, Post Miocardiac Infarction, Dissection, Lesion type, Diameter of Stents, Length of Lesion, Number of Stents and Arterial Hypertension. Many of them are considered as risk factors for heart failure and we are interested to construct a model that determines the impact of these risk factors in the mortality of the patients studied.



2. Bootstrap methodology

Traditional methods for statistical inference often require fulfilling several assumptions and sometimes the measures of accuracy are based on asymptotic theory and cannot be used for small samples. A recent and effective alternative is using the bootstrap method, introduced by Efron (1979). The bootstrap is a computer-intensive resampling method for estimating a population statistics by resampling the data with replacement. The idea behind the bootstrap method is that in the conditions of the lack of any information about the population distribution, the sample contains all the required information and using this information in the right way can lead to good results.

In the traditional methods of statistics, the random sample is used to obtain the statistics of interest. On the other hand, when using bootstrap, we have an empirical distribution and we obtain the bootstrap sample. Then after repeating the process a large number of times, we obtain the bootstrap estimate of the sampling distribution using the Monte Carlo method.

The bootstrap algorithm to obtain a bootstrap estimate and also for estimating the standard error is as follows:

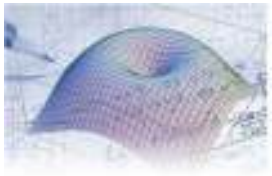
Let we have a sample $X_1, X_2, ..., X_n$ *from a population with unknown distribution F and an unknown parameter* $\boldsymbol{\theta}$.

- *Calculate estimator* $\hat{\theta}$ *for unknown parameter* θ.
- *Construct empirical distribution* $\widehat{F}$.
- *From* $\hat{F}$, *we take the sample* $X_1^*, X_2^*, ..., X_n^*$ *(resample)*
- *Approximate the distribution of* $\hat{\theta}$ *with the distribution of* $\hat{\theta}^*$ *(obtained from bootstrap sample)*
- *Repeat B times obtaining bootstrap estimations* $\hat{\theta}^*_1, \hat{\theta}^*_2, \dots, \hat{\theta}^*_B$.
- *Based on Monte Carlo method, bootstrap estimation for unknown parameter* θ *is:*

$$\hat{\theta}^* = \frac{1}{B}\sum_{b=1}^{B} \hat{\theta}^*_b$$

- *Estimate the standard error of* $\hat{\theta}$ *by the standard deviation of the B replications:*

$$\widehat{se}_B(\hat{\theta}) = [\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}^*_b - \hat{\theta}^*)^2]^{\frac{1}{2}}$$



3. Logistic regression

Multiple logistic regression analysis is used when there is a single dichotomous outcome and more than one independent variable. The independent variables are called covariates and can be continuous or categorical variables. We use multiple logistic regression to construct a model to explain the relationship between the explanatory variables (covariates) and the binary outcome. The outcome as a binary variable it takes two values and often is used 0 and 1 to code the outcomes. Usually 1 is used to denote the outcome of interest in the binary variable. The multiple logistic regression model can be written as follows:

$$\ln(\frac{\pi}{1-\pi}) = a_0 + a_1X_1 + a_2X_2 + \ldots + a_mX_m$$

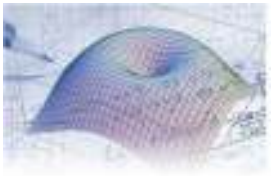
(π is the probability that the outcome is present, 1-π is the probability that the outcome is 0 and we have m covariates).

The logit transformation is called the natural logarithm transformation of the probabilities (the left side in the model). If we want to calculate the probability that the outcome is 1 when a particular value of X is given, we have:

$$\pi = \frac{e^{a_0+a_1X_1+a_2X_2+\ldots+a_mX_m}}{1+a_0+a_1X_1+a_2X_2+\ldots+a_mX_m}.$$

So, the coefficients of the model are $a_0, a_1, a_2, ..., a_m$ and odd ratios are obtained by taking the exponent of the coefficients. To test the significance of the regression coefficients Wald ratio is used. If the significance value from the test is less than 0.05, it indicates that the corresponding coefficient is significant in the model. We can construct 95% confidence intervals for the coefficients of the model and if the confidence interval includes value zero, then the coefficient itself cannot be significant in the model. We can also calculate odd ratios and confidence intervals for odd ratios, taking in consideration that a confidence interval for the odd ratio that includes 1, indicates that the corresponding odd ratio is not significant in the model.

As measures of fit there are Classification Tables that indicate the percentage of correctly predicated values by the given model. Also there is the possibility to use Hosmer and Lemeshow test that gives good results especially in large datasets. This test is a statistical test for goodness of fit for the logistic regression model. In SPSS the Hosmer Lemeshow statistics indicates a poor fit



if the significance value is less than 0.05. The Omnibus test is a likelihood ratio Chi Square test of current model versus the null model and a significance value less than 0.05 indicates that the current model outperform the null model. The Nagelkerke pseudo R^2 indicates an approximated percentage that the variance of the dependent variable is accounted by the predictors overall.

4. R and SPSS

In our study we used both SPSS and R to obtain the results for multiple logistic regression model and also for applying bootstrap.

SPSS (Statistical Package for the Social Sciences) is a statistical package and it is used by researchers to conduct advanced statistical analysis.

R is a programming language and free software environment for statistical computing and graphics.

In this study we compare the performance of both R and SPSS in finding the model and performing bootstrap replications.

RESULTS

As stated above, we used the data records from the Cardiology Department in a hospital in Tirana, Albania, with a total of 2201 patients that include: Gender, Age, Coronary Artery Disease, Diabetes Mellitus, Ejection Fraction, Body Mass Index, Smoking, Dyslipidemia, Post Myocardial Infarction, Dissection, Lesion type, Diameter of Stents, Length of Lesion, Number of Stents and Arterial Hypertension. Many of them are considered as risk factors for heart failure and we are interested to construct a model that determines the impact of these risk factors in Mortality of the patients studied.

Our dependent variable is Mortality, and it is a dichotomous variable. We performed Multiple Logistic Regression to determine which factors are significant in the model, and the impact of each of them in the mortality following the hospitalization for the patients with heart failure.

The dependent variable is mortality (coded 0 and 1), with value 0 that indicates the survivors and 1 indicates the patients that did not survive. There are 15 explanatory variables in the model, but all of them are not significant in the model. We have marked in red the significant variables in the model in the tables below. Using SPSS, we obtained the following results:

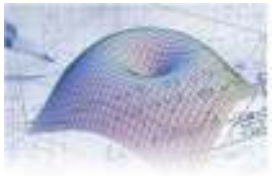


Table 1 Omnibus Tests of Model Coefficients

		Chi-square	df	Sig.
Step 1	Step	105.290	15	.000
	Block	105.290	15	.000
	Model	105.290	15	.000

Table 2 Model summary

Step	-2 Log likelihood	Cox & Snell R Square	Nagelkerke R Square
1	690.030	.047	.154

Table 3 Hosmer and Lemeshow Test

Step	Chi-square	df	Sig.
1	4.075	8	.850

Table 4 Classification Table

	Observed		Predicted		
			Mortality		Percentage Correct
			0	1	
Step 1	Mortality	0	2101	3	99.9
		1	95	2	2.1
	Overall Percentage				95.5

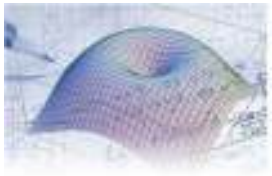


Table 5 Coefficients estimations, odd ratios and 95% confidence intervals for the odd ratios

	Coefficient	S.E.	Sig.	Odd Ratios	95% Confidence Interval for Odd Ratios	
					Lower	Upper
GENDER	-.1339119	.241	.579	.875	.545	1.404
AGE	-.0091754	.015	.542	.991	.962	1.021
DIABETES	-.2954351	.232	.202	.744	.473	1.172
EF	-.5860245	.220	.008	.557	.361	.857
SMOKE	-.0531130	.272	.845	.948	.556	1.617
DISLIPID	.2221214	.285	.436	1.249	.714	2.182
BMI	.6156367	.254	.015	1.851	1.124	3.047
HTA	.9527836	.304	.002	2.593	1.429	4.704
DISEK	-.8244973	.379	.029	.438	.209	.921
DAPT	1.757671	.372	.000	5.799	2.799	12.016
DIAMETER	.061569	.334	.854	1.064	.553	2.045
LENGTH	-.002297	.016	.887	.998	.967	1.030
STNUMBER	.064158	.214	.764	1.066	.701	1.622
LESION	1.87865	.286	.000	6.545	3.737	11.461
PostDil	-.312263	.254	.219	.732	.445	1.204
Constant	-2.96324	1.455	.042	.052		

By applying non-parametric bootstrap using SPSS, with the number of replications N=1000, we obtain the following results:

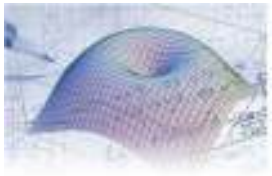


Table 6 Bootstrap for the variables in the equation (coefficients, bias, standard error, significance and the 95 % bootstrap confidence interval for the coefficients)

Bootstrap for Variables in the Equation

	Coefficient	Bootstrap				
		Bias	Std. Error	Sig. (2-tailed)	95% Confidence Interval for Coefficient	
					Lower	Upper
GENDER	-.134	-.022	.244	.549	-.665	.328
AGE	-.009	.000	.017	.580	-.041	.024
DIABETES	-.295	-.003	.242	.208	-.787	.163
EF	-.358	-.009	.221	.009	-1.024	-.146
SMOKE	-.053	-.011	.296	.853	-.673	.489
DISLIPID	.222	-.014	.314	.461	-.408	.817
BMI	.616	.013	.247	.010	.131	1.128
HTA	.953	.012	.353	.005	.209	1.674
DISEK	-.824	.001	.441	.041	-1.806	.142
DAPT	1.758	.010	.409	.001	.892	2.541
DIAMETER	.062	-.030	.307	.815	-.583	.629
LENGTH	-.002	.001	.017	.888	-.036	.030
STNUMBER	.064	-.014	.227	.759	-.405	.501
LEZION	1.879	.056	.345	.001	1.272	2.613
PostDM	-.312	.005	.275	.227	-.856	.284
Constant	-2.963	-.005	1.350	.024	-5.672	-.506

Using R programming language to obtain the Logistic Regression Model, we obtain the following results:

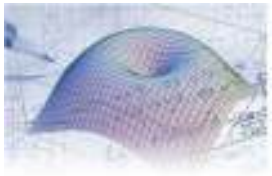


Table 7 Coefficients of the Logistic Regression Model using R

	Coefficient	Sig.	Odd Ratios	95% Confidence Interval for the Odd Ratios	
				Lower	Upper
Constant	-2.963	0.0417	0.5165	0.0029	0.8923
GENDER	-.134	0.579	0.8747	0.5397	1.3939
AGE	-.009	0.542	0.9909	0.9622	1.0207
DIABETES	-.295	0.202	0.7442	0.4736	1.1769
EF	-.586	0.008	0.5565	0.3612	0.859
SMOKE	-.053	0.845	0.9483	0.5477	1.5992
DISLIPID	.222	0.436	1.2487	0.7083	2.1699
BMI	.618	0.0154	1.8508	1.1252	3.0534
HTA	.953	0.00172	2.5929	1.4059	4.6301
PostEM	-0.312	0.219	0.7318	0.4478	1.2157
DISEK	-.824	0.0295	0.4385	0.214	0.952
DAPT	1.758	2.26e-0.6	5.7989	2.7237	11.7939
DIAMETER	.062	0.8536	1.0635	0.5447	2.0159
LENGTH	-.002	0.8871	0.9977	0.9663	1.0297
STNUMBER	.064	0.764	1.0663	0.6946	1.6099
LESION	1.879	4.98e-11	6.5447	3.7987	11.6938

Table 8 R squares from Cox and Snell and Nagelkerke

Cox & Snell R Square	Nagelkerke R Square
.047	.154

We use the nonparametric bootstrap in R with the number of bootstrap replications N=1000, and we have the results:

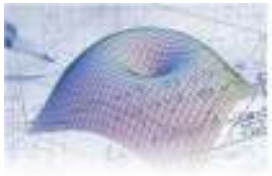


Table 9 Bootstrap for the variables in the equation (coefficient, bias and standard error)

	Coefficient	Bias	Standard error
Constant	-2.963	-0.7745	1.29243
GENDER	-.134	-0.00315	0.2511
AGE	-.009	-0.00021	0.01658
DIABETES	-.295	0.00584	0.2461
EF	-.586	-0.00119	0.2273
SMOKE	-.053	-0.02497	0.3001
DISLIPID	.222	0.003742	0.30215
BMI	.616	0.01811	0.2583
HTA	.953	-0.00809	0.3405
PostMI	-0.312	0.00506	0.2678
DISEK	-.824	0.0484	0.46199
DAPT	1.758	0.019	0.4188
DIAMETER	.062	-0.0154	0.30206
LENGTH	-.002	-0.00135	0.01693
STNUMBER	.064	0.00441	0.22459
LESION	1.879	0.05701	0.3479

Based on the results obtained, we noticed that we have the same significant factors from both SPSS and R. We have six significant factors that are: Ejection Fraction (EF), Body Mass Index (BMI), Arterial Hypertension (HTA), Dissection (DISEK), Dual Antiplatelet Therapy (DAPT) and Lesion (B2C) (this is determined by the value of sig. in the corresponding column in each table, a value less than 0.05 indicates a significant factor in the model). Factors that increase the odds to the outcome are Body Mass Index (BMI), Arterial Hypertension (HTA), Dual Antiplatelet Therapy (DAPT) and Lesion (B2C). Factor that decrease the odds to the outcome are Ejection Fraction (EF) and Dissection (DISEK).

We have the same model coefficients and confidence intervals close to each other using both R and SPSS.

Applying bootstrap resulted in good estimations from both R and SPSS, but to highlight the similarities and differences between R and SPSS, we have constructed graphs to make a more evident comparison:

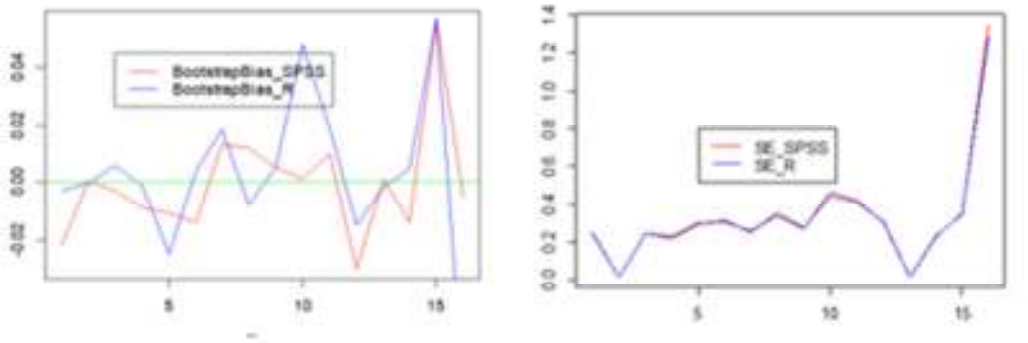


Figure 1 Comparison of Bias and standard error from bootstrap estimations using R and SPSS

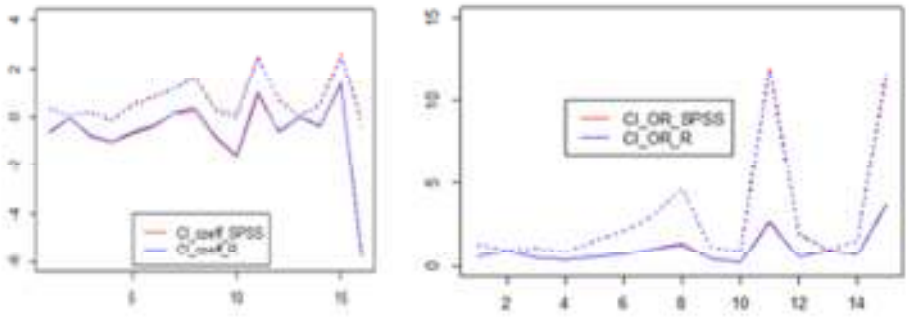


Figure 2 Confidence Intervals for coefficients and the odd ratios of logistic regression model using R and SPSS

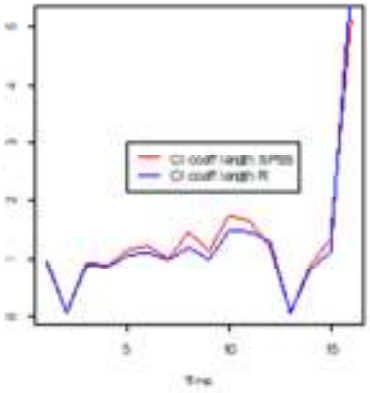


Figure 3 Comparing interval length for model coefficients using R and SPSS

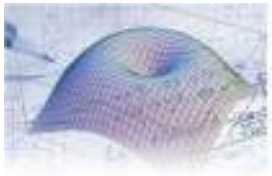
CONCLUSIONS

To identify the factors that predict mortality in the in a data records sample from the hospital that have cardiac problems, a multiple logistic regression was conducted, entering 15 covariates in the model. The results indicated that we have only six significant factors in the model.

The Omnibus Tests of Model Coefficients indicates a significant model also the Hosmer and Lemeshow Test indicates that the model fits the data.

The overall percentage must be at least 80% (in our model this value is 95.5%) so this is a good indicator for the model obtained. We had the same results from both SPSS and R in determining the significant factors for the model: Ejection Fraction (EF), Body Mass Index (BMI), Arterial Hypertension (HTA), Dissection (DISEK), Dual Antiplatelet Therapy (DAPT), and Lesion (B2C). The impact of these factors in predicting the mortality and the risk associated with each of them is different.

Factors that increase the odds to the outcome are Body Mass Index (BMI), Arterial Hypertension (HTA), Dual Antiplatelet Therapy (DAPT) and Lesion (B2C). The factors that decrease the odds to the outcome are Ejection Fraction (EF) and Dissection (DISEK).



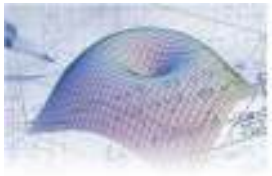
Also model coefficients and odd ratios are the same from both R and SPSS. Bootstrap gives good results in both cases, but we noticed that bias obtained from R is smaller than the bias obtained using SPSS and the bootstrap Standard Error is nearly the same in both cases. We calculated Confidence Interval length for the coefficients and in most of cases we have smaller interval length using SPSS than R.

BIBLIOGRAPHY

1. B. Efron: “Bootstrap methods: another look at the Jackknife" Annals of Statistics, 7, pp 1- 26. (1979)
2. B. Efron, R. J. Tibshirani: An Introduction to the Bootstrap. New York: Chapman and Hall. (1993)
3. C. Davidson, D.V. Hinkley: Bootstrap Methods and their Application. Cambridge: Cambridge University Press. (1997)
4. Ahmed Hossain H. T. Abdullah, Nonparametric bootstrapping for multiple logistic regression model using R, BRAC University Journal, vol. I, no. 2, 2004, pp. 109-113.
5. Adjei, I and Karim, R. (2016), An Application of Bootstrapping in Logistic Regression Model, Open Access Library Journal, 3, 1-9, doi: 10.4236/oalib.1103049.
6. Newsom Psy 525/625 Categorical Data Analysis, Spring 2021.

www.Web.pdx.edu
www.ibm.com
https://www.r-bloggers.com/2015/08/bootstrap-with-logistic-regression/
https://online.stat.psu.edu/stat462/node/207/
http://sahirbhatnagar.com/
http://www.m-hikari.com/ams/ams-2017/ams-9-12-2017/p/margoAMS9-12-2017
http://galton.uchicago.edu/~eichler/stat24600/Handouts/bootstrap.pdf
www.sciencedirect.com/topics/computer-science/bootstrap-method

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



N-Tuple Weak Orbits Tending to Infinity for Banach Space Operators

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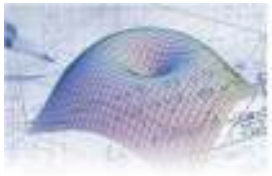
Abstract: In this paper we prove some results on the existence of a dense set of pairs in the product of an infinite-dimensional complex Banach space with its dual space such that each pair of this set has an n −tuple weak orbit tending to infinity for specific countable family of mutually commuting bounded linear operators.

Keywords: Banach spaces, weak orbits, *n*-tuple weak orbits, sequences of operators.

1. Introduction

Let X be an infinite-dimensional Banach space over the field of complex numbers $\mathbb{C}$, $B(X)$ the algebra of all bounded linear operators on X and X^* its dual space, i.e., the space of all bounded linear functionals $x^*: X \to \mathbb{C}$. For the direct product $X \times X^*$ we assume that is endowed with the product topology. As usually, if $x \in X$ and $x^* \in X^*$, we will denote $\langle x, x^* \rangle := x^*(x)$. If $T \in B(X)$, then $\sigma(T)$ and $r(T)$ will denote the spectrum and the spectral radius of the operator T, respectively.

If $T_1, T_2, \dots, T_n \in B(X)$ are mutually commuting operators, n −*tuple orbit* of the vector $x \in X$ (or, the *orbit of* x under the n −tuple $\mathbf{T} = (T_1, T_2, \dots, T_n)$) is the set



$$\mathrm{Orb}(\{T_i\}_{i=1}^n, x) = \left\{T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x: k_i \geq 0; 1 \leq i \leq n\right\}. \quad (1.1)$$

The n −tuple orbit *tends to infinity* if

$$\lim_{k_i\to\infty}\left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n = 1$, the n −tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T, x) = \{T^n x: n = 0,1,2,\dots\},$$

usually referred as *single orbit* (or, simply *orbit*) of the vector $x \in X$ under the operator T.

The n −*tuple weak orbit* of the pair $(x, x^*) \in X \times X^*$ (or, the *weak orbit of the pair* (x, x^*) under the n −tuple $\mathbf{T} = (T_1, T_2, \dots, T_n)$) is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*) = \left\{\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle: k_i \geq 0; 1 \leq i \leq n\right\}. \quad (1.2)$$

The n −tuple weak orbit *tends to infinity* if

$$\lim_{k_i\to\infty}\left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n = 1$, the n −tuple weak orbit (1.2) reduces to a sequence of form

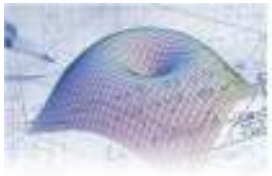
$$\mathrm{Orb}(T, x, x^*) = \{\langle T^n x, x^*\rangle: n = 0,1,2,\dots\},$$

usually referred as *weak orbit* of the pair $(x, x^*) \in X \times X^*$ under the operator T.

In [5] we gave only a brief survey without proofs of some results on the existence of a dense set of pairs $(x, x^*) \in X \times X^*$ each having a single weak orbit tending to infinity under every operator of a specific sequence of operators in $B(X)$. In this paper we are going to give an appropriate generalization for n −tuple weak orbits only of the results in [5] that do not involve any requirements upon specific subsets of the spectra of the operators.

2. Preliminary Results

Lemma 2.1. ([2, Lemma V.37.15]) *Let* $\varepsilon > 0$ *and* $(a_n)_{n\geq 1}$ *be a sequence of positive numbers satisfying* $\sum_{n\geq 1} a_n < \varepsilon$. *Then there is a sequence of positive numbers* $(b_n)_{n\geq 1}$ *such that* $b_n \to \infty$ *as* $n \to \infty$ *and* $\sum_{n\geq 1} a_n b_n < \varepsilon$.



Theorem 2.2. ([2, Theorem V.39.5]) *Let X and Y be Banach spaces and $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$. Let $(a_n)_{n\geq 1}$ be sequence of positive numbers with $\sum_{n\geq 1} a_n^{1/2} < \infty$. Then there are $x \in X$ and $y^* \in Y^*$ such that*

$$|\langle T_n x, y^*\rangle| \geq a_n \|T_n\|, \text{ for all } n \geq 1.$$

Moreover, given balls $B \subset X$ and $B^* \subset Y^*$ of radii greater than $\sum_{n\geq 1} a_n^{1/2} < \infty$, then it is possible to find $x \in B$ and $y^* \in B^*$ with this property.

Corollary 2.3. ([2, Corollary V.39.6]) *Let X and Y be Banach spaces and $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$ satisfying $\sum_{n=1}^{\infty} \|T_n\|^{-1/2} < \infty$. Then there exist $x \in X$ and $y^* \in Y^*$ such that $|\langle T_n x, y^*\rangle| \to \infty$. Moreover, the set of such pairs (x, y^*) is dense in $X \times Y^*$.*

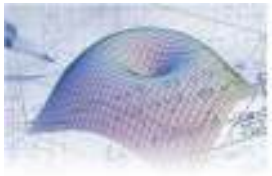
3. Main Results for N-Tuple Weak Orbits

Let $F = \{1,2,\dots,N\}$ for some $N \in \mathbb{N}$, $N \geq 2$, or $F = \mathbb{N}$.

Theorem 3.1. *If X is a Banach space, $\{T_i : i \in F\} \subset B(X)$ and $\left\{\left(a_{i,j}\right)_{j\geq 1} : i \in F\right\}$ is a family of sequences of positive numbers such that $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2} < \infty$, then for every open ball $B \subset X$ and every open ball $B^* \subset X^*$ with radii strictly larger then $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2}$ there are $x \in B$ and $x^* \in B^*$ such that*

$$\left|\langle T_i^k x, x^*\rangle\right| \geq a_{i,k} \left\|T_i^k\right\|, \text{ for all } i \in F \text{ and } k \in \mathbb{N}.$$

Proof. Let $B \subset X$ and $B^* \subset X^*$ be open balls, each with radius strictly larger then $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2}$ and put $T_{i,k} \coloneqq T_i^k$, $i \in F$, $k \in \mathbb{N}$. Let $f : F \times \mathbb{N} \to \mathbb{N}$ be the bijective mapping defined with:



$$f(i,j)=\begin{cases} i+N(j-1), & \text{if } F=\{1,2,\dots,N\},\\ \dfrac{(i+j-2)(i+j-1)}{2}, & \text{if } F=\mathbb{N}, \end{cases}$$

and $g:\mathbb{N}\to F\times\mathbb{N}$ denote its inverse. If $(\alpha_n)_{n\geq 1}$ is a sequence of positive numbers and $(S_n)_{n\geq 1}$ is a sequence of operators defined with

$$\alpha_n = a_{g(n)} \text{ and } S_n = T_{g(n)}, \text{ for all } n\in\mathbb{N},$$

then $\sum_{n\geq 1}\alpha_n^{1/2}=\sum_{i\in F,j\geq 1}a_{i,j}^{1/2}<\infty$ and hence (by Theorem 2.2, applied on $(\alpha_n)_{n\geq 1}$, $(S_n)_{n\geq 1}$ and $X=Y$) there are $x\in B$ and $x^*\in B^*$ such that

$$|\langle S_n x,x^*\rangle|\geq \alpha_n\|S_n\|, \text{ for all } n\geq 1.$$

Given $(i,k)\in F\times\mathbb{N}$ and $n=f(i,k)$, these inequalities, along with the definition of $(\alpha_n)_{n\geq 1}$ and $(S_n)_{n\geq 1}$, will give

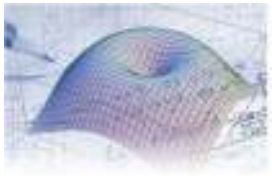
$$\left|\langle T_i^k x,x^*\rangle\right|=\left|\langle T_{i,k}x,x^*\rangle\right|=\left|\langle T_{g(n)}x,x^*\rangle\right|\geq a_{g(n)}\left\|T_{g(n)}\right\|=a_{i,k}\left\|T_i^k\right\|,$$

which completes the proof. ■

Theorem 3.2. *If X is Banach space and $\{T_i: i\in F\}\subset B(X)$ is a family of operators such that $\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-1/2}<\infty$, for all $i\in F$, then there is a dense set $D\subset X\times X^*$ such that the weak orbit $\mathrm{Orb}(T_i,x,x^*)$ tends to infinity for every pair $(x,x^*)\in D$ and every $i\in F$. If, in addition, $\{T_i: i\in F\}$ is a family of mutually commuting operators such that the sequences $(T_i^k-T_j^k)_{k\geq 1}$ are norm bounded for all $i,j\in F$, then for every $n\in F$, every $1<m\leq n$ and every pair $(x,x^*)\in D$, the $m-$tuple weak orbit $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^m,x,x^*\right)$ tends to infinity for all $1\leq i_1<i_2<\cdots<i_m\leq n$.*

Proof. To prove the first assertion let $\varepsilon>0$, $B\subset X$ and $B^*\subset X^*$ be open balls each with a radius ε. For $i\in F$, let $\varepsilon_i>0$ be such that

$$\varepsilon_i\left(\sum_{k=1}^{\infty}\frac{1}{\left\|T_i^k\right\|^{1/2}}\right)<\frac{\varepsilon}{2^{i+1}},$$



and, by Lemma 2.1, let $(b_{i,k})_{k\geq 1}$ be the sequence of positive numbers such that $b_{i,k} \to \infty$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^{1/2}} < \frac{\varepsilon}{2^{i+1}}. \tag{3.1}$$

For $(i,k) \in F \times \mathbb{N}$, put $a_{i,k} = \varepsilon_i^2 b_{i,k}^2 \left\|T_i^k\right\|^{-1}$. Then by (3.1) we have

$$\sum_{i\in F, k\geq 1} a_{i,k}^{1/2} = \sum_{i\in F} \sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^{1/2}} < \sum_{i\in F} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}.$$

Hence, by Theorem 3.1, there are $x \in B$ and $x^* \in B^*$ such that

$$\left|\langle T_i^k x, x^*\rangle\right| \geq a_{i,k}\left\|T_i^k\right\| = \varepsilon_i^2 b_{i,k}^2 \left\|T_i^k\right\|^{-1}\left\|T_i^k\right\| = \varepsilon_i^2 b_{i,k}^2, \text{ for all } i \in F \text{ and } k \geq 1.$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty}\left|\langle T_i^k x, x^*\rangle\right| = \infty, \text{ for all } i \in F. \tag{3.2}$$

If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k\geq 1}$ is norm bounded for all $i, j \in F$, let $M_{i,j} > 0$ be such that $\left\|T_i^k - T_j^k\right\| \leq M_{i,j}$, for all $k \geq 0$, and let $(x, x^*) \in X \times X^*$ be a pair satisfying (3.2). We continue by induction.

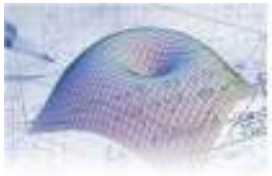
Let $m = 2$ and $1 \leq i_1 < i_2 \leq n$. Then

$$\begin{aligned}
\left|\langle T_{i_1}^{k_1+k_2} x, x^*\rangle\right| &\leq \left|\langle T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&= \left|\langle T_{i_1}^{k_1}\left(T_{i_1}^{k_2} - T_{i_2}^{k_2}\right)x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&\leq \left\|T_{i_1}^{k_1}\right\| \cdot \left\|T_{i_1}^{k_2} - T_{i_2}^{k_2}\right\| \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&\leq \left\|T_{i_1}\right\|^{k_1} \cdot M_{i_1,i_2} \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right|.
\end{aligned}$$

Since $\left|\langle T_{i_1}^n x, x^*\rangle\right| \to \infty$ as $n \to \infty$ (hence $\left|\langle T_{i_1}^{k_1+k_2} x, x^*\rangle\right| \to \infty$ as $k_2 \to \infty$, for all $k_1 \geq 0$), the above inequalities imply that

$$\left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \to \infty, \text{ as } k_2 \to \infty, \text{ for all } k_1 \geq 0.$$

Similarly,



$$\begin{aligned}\left|\langle T_{i_2}^{k_1+k_2}x, x^*\rangle\right| &\le \left|\langle T_{i_2}^{k_1+k_2}x - T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right| \\ &= \left|\langle T_{i_2}^{k_2}\left(T_{i_2}^{k_1} - T_{i_1}^{k_1}\right)x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right| \\ &\le \left\|T_{i_2}^{k_2}\right\| \cdot \left\|T_{i_2}^{k_1} - T_{i_1}^{k_1}\right\| \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right| \\ &\le \left\|T_{i_2}\right\|^{k_2} \cdot M_{i_2,i_1} \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right|,\end{aligned}$$

imply that

$$\left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x, x^*\rangle\right| \to \infty, \text{ as } k_1 \to \infty, \text{ for all } k_2 \ge 0.$$

To complete the proof, it is enough to show that the claim is true for $m = n$, under an inductive assumption that the $(n-1)$ −tuple weak orbit $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^{n-1}, x, x^*\right)$ tends to infinity for all $1 \le i_1 < i_2 < \cdots < i_{n-1} \le n$. For fixed $i \in \{1,2,\dots,n\}$ and $k_1,\ k_2, \dots,\ k_n \ge 0$ and arbitrary $j \in \{1,2,\dots,n\} \setminus \{i\}$, we have

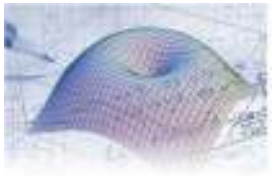
$$\begin{aligned}&\left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x, x^*\rangle\right| \\ &\le \left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x - T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right| \\ &+ \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right| \\ &= \left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}\left(T_j^{k_i} - T_i^{k_i}\right)x, x^*\rangle\right| + \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right| \\ &\le \left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}\right\| \cdot \left\|T_j^{k_i} - T_i^{k_i}\right\| \cdot \|x\| \cdot \|x^*\| \\ &+ \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right| \\ &\le \left(\prod_{\substack{p=1 \\ p\ne i}}^{n}\left\|T_p\right\|^{k_p}\right) \cdot M_{j,i} \cdot \|x\| \cdot \|x^*\| + \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x, x^*\rangle\right|.\end{aligned}$$

Since $j \in \{1,2,\dots,n\} \setminus \{i\}$,

$$\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x, x^*\rangle \in Orb(\{T_1,\dots,T_{i-1},T_{i+1},\dots,T_n\}, x, x^*),$$

and since, by the inductive assumption, this $(n-1)$ −tuple weak orbit tents to infinity, we have

$$\left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x, x^*\rangle\right| \to \infty, \text{ as } k_i \to \infty, \text{ for all } k_j \ge 0,\ j \ne i.$$



This, together with the above inequalities implies that

$$\left|\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \rangle\right| \to \infty, \text{ as } k_i \to \infty, \text{ for all } k_j \geq 0, j \neq i.$$

which completes the proof. ■

Corollary 3.3. *If X is Banach space and $\{T_i : i \in F\} \subset B(X)$ is a family of operators such that $r(T_i) > 1$ for all $i \in F$, then there is a dense set $D \subset X \times X^*$ such that the weak orbit $\mathrm{Orb}(T_i, x, x^*)$ tends to infinity for every pair $(x, x^*) \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequences $(T_i^k - T_j^k)_{k \geq 1}$ are norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$ and every pair $(x, x^*) \in D$, the m −tuple weak orbit $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^m, x, x^*\right)$ tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.*

Proof. By Theorem 3.2 it is enough to show that an operator $T \in B(X)$ with spectral radius $r(T) > 1$ satisfies: $\sum_{n=1}^{\infty} \|T^n\|^{-1/2} < \infty$. If $r(T) > 1$, then there is $\lambda \in \sigma(T)$ such that $|\lambda| > 1$. Clearly $|\lambda|^{1/2} > 1$ and hence the series $\sum_{n=1}^{\infty} |\lambda|^{-n/2}$ converges. On the other hand, by the Spectral Mapping Theorem, $\lambda^n \in \sigma(T^n)$ for all $n \in \mathbb{N}$. Hence $|\lambda|^n \leq r(T^n) \leq \|T^n\|$ and $\sum_{n=1}^{\infty} \|T^n\|^{-1/2} \leq \sum_{n=1}^{\infty} |\lambda|^{-n/2} < \infty$. ■

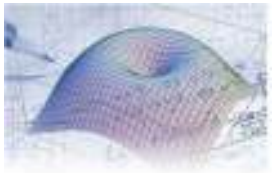
4. Few Remarks on N-Tuple Orbits Tending to Infinity

The inequalities of form

$$\left|\langle T_1^{k_1} \dots T_n^{k_n} x, x^* \rangle\right| \leq \left\|T_1^{k_1} \dots T_n^{k_n} x\right\| \cdot \|x^*\|, (x, x^*) \in X \times X^*, k_i \geq 0, 1 \leq i \leq n,$$

clearly imply that the n −tuple orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity whenever there is $x^* \in X^*$ such that the n −tuple weak orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*)$ tends to infinity. Hence, in the light of the results from the previous section, we can state the following alternative results for n −tuple orbits tending to infinity to a part of the results in [7].

Theorem 4.1. *If X is Banach space and $\{T_i : i \in F\} \subset B(X)$ is a family of operators such that $\sum_{k=1}^{\infty} \left\|T_i^k\right\|^{-1/2} < \infty$, for all $i \in F$, then there is a dense set $D \subset X$ such that the orbit*



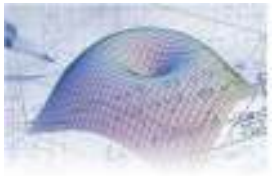
$\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and every* $i \in F$*. If, in addition,* $\{T_i : i \in F\} \subset B(X)$ *is a family of mutually commuting operators such that the sequences* $(T_i^k - T_j^k)_{k\geq 1}$ *are norm bounded for all* $i, j \in F$*, then for every* $n \in F$*, every* $1 < m \leq n$ *and every* $x \in D$*, the* $m-$*tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ *tends to infinity for all* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$*.*

Corollary 4.2. *If* X *is Banach space and* $\{T_i : i \in F\} \subset B(X)$ *is a family of operators such that* $r(T_i) > 1$ *for all* $i \in F$*, then there is a dense set* $D \subset X$ *such that the orbit* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and every* $i \in F$*. If, in addition,* $\{T_i : i \in F\} \subset B(X)$ *is a family of mutually commuting operators such that the sequences* $(T_i^k - T_j^k)_{k\geq 1}$ *are norm bounded for all* $i, j \in F$*, then for every* $n \in F$*, every* $1 < m \leq n$ *and every* $x \in D$ *the* $m-$*tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ *tends to infinity for all* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$*.*

References:

[1] S. Mančevska, M. Orovčanec, "Orbits tending to infinity under sequences of operators on Banach spaces II", Math. Maced., Vol. 5 (2007), 57–61.
[2] V. Müller, "Spectral theory of linear operators and spectral systems in Banach algebras", (2nd ed.), Operator Theory: Advances and Applications Vol. 139, Birkhäuser Verlag AG, Basel - Boston - Berlin, 2007.
[3] V. Müller, J. Vršovský, "Orbits of linear operators tending to infinity", Rocky Mountain J. Math., Vol. 39 No.1 (2009), 219–230.
[4] S. Mančevska, M. Orovčanec, "Orbits tending to infinity under sequences of operators on Banach spaces", International Journal of Pure and Applied Mathematics 47(2) (2008), 175–183.
[5] S. Mančevska, M. Orovčanec, "Few results on weak orbits under sequences of operators", Proceedings of the Fourth International Scientific Conference – FMNS2011, Vol.2, (2011) p.33-40.
[6] A. Tajmouati, Y. Zahouan, "Orbit of tuple of operators tending to infinity", International Journal of Pure and Applied Mathematics Vol. 110, No.4 (2016), 651–656.
[7] S. Mančevska, M. Orovčanec, "N-Tuple orbits tending to infinity", Proceedings of the CODEMA 2020, (2020) p.23-31.

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



New integral inequalities for exponentially nonconvex functions pertaining Raina function

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Abstract: In this paper, the authors defined a new general class of functions, the so-called exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex function involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function). Utilizing this, some Hermite–Hadamard type integral inequalities via generalized fractional integral operator are obtained. Some new results as special cases are given as well.

1. Introduction and preliminaries

Convexity is one of the fascinating and natural concepts, it is beneficial in optimization theory, theory of inequalities, numerical analysis, economics and in other subjects of pure and applied mathematics. Convex functions are defined in different ways due to their interesting graphical shapes in euclidean space. A convex function defined on an interval of real line is always continuous in the interior points, but need not be differentiable. Although, it has left and right increasing derivatives at each interior point. The derivative of a differentiable convex function is always an increasing function. A twice differentiable convex function has downward concavity. In analytical forms it is defined in several ways and the classical one is given in [1] as follows:

Definition 1.1. A function $\psi: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I, if

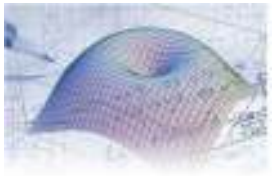
$$\psi(\imath\xi_1 + (1-\imath)\xi_2) \leq \imath\psi(\xi_1) + (1-\imath)\psi(\xi_2)$$

holds for all $\xi_1, \xi_2 \in I$ and $\imath \in [0,1]$.

In [2], the notion of h–convex function is introduced.

Definition 1.2. Let $h: [0,1] \to [0,+\infty)$ be a function. A function $\psi: I \to \mathbb{R}$ is said to be h–convex, if

$$\psi(\imath\xi_1 + (1-\imath)\xi_2) \leq h(\imath)\psi(\xi_1) + h(1-\imath)\psi(\xi_2)$$



holds for every $\xi_1, \xi_2 \in I$ and $\imath \in [0,1]$.

Exponentially convex functions have been introduced by Awan et al., see [3].

Definition 1.3. A function $\psi: I \subseteq \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if

$$\psi(\imath\xi_1 + (1-\imath)\xi_2) \leq \imath\frac{\psi(\xi_1)}{e^{\omega\xi_1}} + (1-\imath)\frac{\psi(\xi_2)}{e^{\omega\xi_2}}$$

holds for all $\xi_1, \xi_2 \in I$, $\omega \in \mathbb{R}$ and $\imath \in [0,1]$.

Exponentially convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

The most significant inequality is the Hermite–Hadamard integral inequality, see [4]. This double inequality is expressed as:

$$\psi\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(\imath)d\imath \leq \frac{\psi(\xi_1)+\psi(\xi_2)}{\mathbf{2}}. \tag{1.1}$$

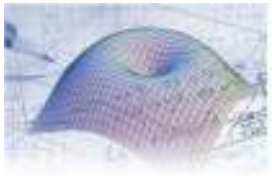
The double inequality (1.1) became a very important foundation within the field of mathematical analysis and optimization, several applications of these inequalities have been found in number of settings. Furthermore, several inequalities of special means can be discovered for the specific options of the function ψ. Due to large applications of double inequality (1.1), literature is growing and giving its some new proofs, augmentations, improvements and generalizations, see $[5-14]$ and the references therein.

In [15], Raina R. K. introduced a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(z) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots}(z) = \sum_{k=0}^{+\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}z^k, \tag{1.2}$$

where $\rho, \lambda > 0, |z| < R$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ is a bounded sequence of positive real numbers. Note that, if we take in (1.2) $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{((\alpha)_k(\beta)_k)}{(\gamma)_k}$ for $k = 0,1,2, \dots$, where α, β and γ are parameters which

can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \dots$), and the symbol $(a)_k$ denote the quantity



$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1), \quad k = 0,1,2,\dots,$$

and restrict its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(z) = F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!\,(\gamma)_k} z^k.$$

Also, if $\sigma = (1,1,\dots)$ with $\rho = \alpha, (Re(\alpha) > 0), \lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in (1.2), then we have the classical Mittag–Leffler function

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

Now we are able to define a new general class of function involving $\mathcal{F}_{\rho,\lambda}^{\sigma}(\cdot)$ (Raina function).

Definition 1.4. Let $h_1, h_2 : [0,1] \to [0,+\infty)$ be two functions and $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$. If ψ satisfies the following inequality

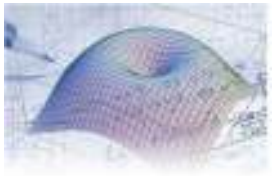
$$\psi\left(\xi_1 + \imath \mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2 - \xi_1)\right) \leq h_1(\imath)\frac{\psi(\xi_1)}{e^{\omega_1 \xi_1}} + h_2(\imath)\frac{\psi(\xi_2)}{e^{\omega_2 \xi_2}},$$

for all $\imath \in [0,1]$, $\omega_1, \omega_2 \in \mathbb{R}$, and $\xi_1, \xi_2 \in I$ where $\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2 - \xi_1) > 0$, then ψ is called exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex function.

Remark 1.1. Taking $\omega_1 = \omega_2 = \omega$, $h_1(\imath) = 1 - \imath$, $h_2(\imath) = \imath$ and $\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2 - \xi_1) = \xi_2 - \xi_1 > 0$ in our definition, then we obtain Definition 1.3.

Remark 1.2. Choosing $\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2 - \xi_1) = \xi_2 - \xi_1 > 0$, let us discuss some special cases in our Definition 1.4 as follows:
(I) Taking $h_1(\imath) = h_2(\imath) = 1$, we have exponentially (ω_1, ω_2, P)–convex function.
(II) Choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get exponentially (ω_1, ω_2, h)–convex function.

(III) Taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain exponentially (s, ω_1, ω_2)–Breckner–convex function.



(IV) Choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have exponentially (s, ω_1, ω_2)–Godunova–Levin–Dragomir–convex function.

(V) Taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get exponentially $(\omega_1, \omega_2, \text{tgs})$–convex function.

(VI) Choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain exponentially $(\omega_1, \omega_2, \text{MT})$–convex function.

To the best of our knowledge these classes of functions are all new.

Definition 1.5. [16] The left and right side generalized fractional integrals for a function ψ are defined as

$$ {}_{\xi_1^+}\mathrm{I}_{\phi}\psi(\tau) = \int_{\xi_1}^{\tau} \frac{\phi(\tau-\imath)}{\tau-\imath}\psi(\imath)d\imath, \quad \tau > \xi_1, $$

$$ {}_{\xi_2^-}\mathrm{I}_{\phi}\psi(\tau) = \int_{\tau}^{\xi_2} \frac{\phi(\imath-\tau)}{\imath-\tau}\psi(\imath)d\imath, \quad \tau < \xi_2. $$

Our main goal during this paper is to prove some Hermite–Hadamard type integral inequalities for exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex functions by using the generalized fractional integral operator. Moreover, we will derive several special cases from our main results. At the end, a briefly conclusion will be given as well.

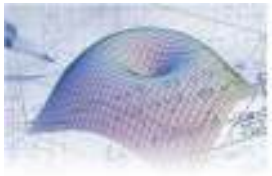
2. Main results

Throughout this section the following notation is used:

$$ \Delta := [\xi_1, \xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)] \quad \text{where} \quad \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1) > 0 $$

and $\omega_1, \omega_2, \varpi_1, \varpi_2 \in \mathbb{R}$.

Theorem 2.1. Let $h_1, h_2 : [0,1] \to [0, +\infty)$ be two continuous functions and $\psi : \Delta \to (0, +\infty)$ an exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex function. Then the following inequality for the generalized fractional integral holds:



(2.1) $$ {}_{\xi_1^+}\mathrm{I}_{\phi}\psi\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)+{}_{\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)^-}\mathrm{I}_{\phi}\psi(\xi_1) \leq \frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}\mathrm{H}_1+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\mathrm{H}_2, $$

where

$$ \mathrm{H}_i := \int_0^1 [\,\mathrm{h}_i(\imath)+\mathrm{h}_i(1-\imath)]\frac{\phi\left(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath\right)}{\imath}\,d\imath, \quad \forall\, i=1,2. $$

Proof. Since ψ is exponentially $(\omega_1, \omega_2, \mathrm{h}_1, \mathrm{h}_2)$–nonconvex function, we have

(2.2) $$ \psi(\xi_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \leq \mathrm{h}_1(\imath)\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\mathrm{h}_2(\imath)\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}} $$

and

(2.3) $$ \psi(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \leq \mathrm{h}_1(1-\imath)\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\mathrm{h}_2(1-\imath)\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}. $$

Adding (2.2) and (2.3), we get

$$ \psi\left(\xi_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)+\psi\left(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right) $$

$$ \leq \left(\mathrm{h}_1(\imath)\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\mathrm{h}_2(\imath)\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right)+\left(\mathrm{h}_1(1-\imath)\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\mathrm{h}_2(1-\imath)\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right) $$

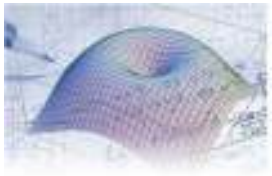
(2.4) $$ = \frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}[\mathrm{h}_1(\imath)+\mathrm{h}_1(1-\imath)]+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}[\mathrm{h}_2(\imath)+\mathrm{h}_2(1-\imath)]. $$

Multiplying (2.4) with $\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath)}{\imath}$ on both sides and integrating the resultant inequality with respect to $\imath$ over $[0,1]$, we obtain our required result (2.1).

We point out some special cases of Theorem 2.1.

Case 1. Taking $\phi(\imath)=\imath$, we obtain the following inequalities for the Riemann integral:

Corollary 2.1. Under assumptions of Theorem 2.1, taking $\mathrm{h}_1(\imath)=\mathrm{h}_2(\imath)=1$, we have



$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le \frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}. \tag{2.5}$$

Corollary 2.2. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le A\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right], \tag{2.6}$$

where

$$A := \int_0^1 h(\imath)d\imath.$$

Corollary 2.3. Under assumptions of Theorem 2.1, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

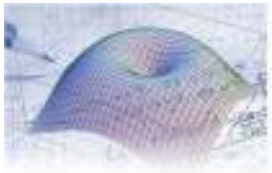
$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le \frac{\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}}{s+1}. \tag{2.7}$$

Corollary 2.4. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le \frac{\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}}{1-s}. \tag{2.8}$$

Corollary 2.5. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le \frac{\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}}{6}. \tag{2.9}$$



Corollary 2.6. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right]^2}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\psi(\tau)\,d\tau \le \frac{\pi\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]}{4}. \tag{2.10}$$

Case 2. Taking $\phi(\imath) = \frac{\imath^{\alpha}}{\Gamma(\alpha)}$, we obtain the following inequalities for the Riemann fractional integral:

Corollary 2.7. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\begin{aligned}&\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]\\&\le\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}.\end{aligned} \tag{2.11}$$

Corollary 2.8. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

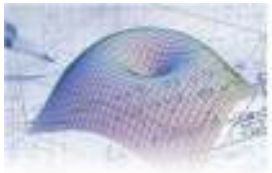
$$\begin{aligned}&\frac{\Gamma^2(\alpha)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]\\&\le H^*\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],\end{aligned} \tag{2.12}$$

where

$$H^* := \int_0^1 \imath^{\alpha-1}\left[h(\imath)+h(1-\imath)\right]d\imath.$$

Corollary 2.9. Under assumptions of Theorem 2.1, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\begin{aligned}&\frac{\Gamma^2(\alpha)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]\\&\le\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]\left(\beta(\alpha,s+1)+\frac{1}{s+\alpha}\right).\end{aligned} \tag{2.13}$$



Corollary 2.10. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{\boldsymbol{\Gamma}^2(\alpha)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]$$

$$\leq\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]\left(\beta(\alpha,1-s)+\frac{1}{\alpha-s}\right). \tag{2.14}$$

Corollary 2.11. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\frac{\boldsymbol{\Gamma}^2(\alpha)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]$$

$$\leq\frac{\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}}{(\alpha+1)(\alpha+2)}. \tag{2.15}$$

Corollary 2.12. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$\frac{\boldsymbol{\Gamma}^2(\alpha)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2\alpha}}\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{\xi_2^-}\psi(\xi_1)\right]$$

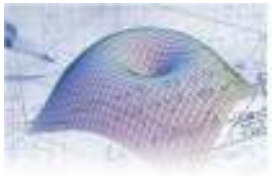
$$\leq\left[\beta\left(\alpha+\frac{1}{2},\frac{1}{2}\right)+\beta\left(\alpha-\frac{1}{2},\frac{3}{2}\right)\right]\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]. \tag{2.16}$$

Case 3. Taking $\phi(\imath) = \frac{\imath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we obtain the following inequalities for the k–Riemann fractional integral:

Corollary 2.13. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\frac{k^2\boldsymbol{\Gamma}_k(\alpha)\boldsymbol{\Gamma}_k(\alpha+k)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\xi_1^+,k}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi(\xi_1)\right]$$

$$\leq\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}. \tag{2.17}$$



Corollary 2.14. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

$$\frac{k^2\Gamma_k^2(\alpha)}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I_{\xi_1^+,k}^{\alpha}\psi(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))+I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-,k}^{\alpha}\psi(\xi_1)\right]$$

$$\leq T^*\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right], \tag{2.18}$$

where

$$T^* := \int_0^1 \imath^{\frac{\alpha}{k}-1}\,[h(\imath)+h(1-\imath)]d\imath.$$

Corollary 2.15. Under assumptions of Theorem 2.1, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\frac{k^2\Gamma_k^2(\alpha)}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I_{\xi_1^+,k}^{\alpha}\psi(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))+I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-,k}^{\alpha}\psi(\xi_1)\right]$$

$$\leq \left[\beta\left(\frac{\alpha}{k},s+1\right)+\frac{k}{ks+\alpha}\right]\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]. \tag{2.19}$$

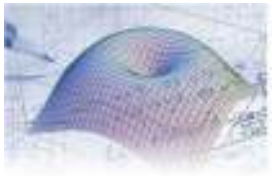
Corollary 2.16. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{k^2\Gamma_k^2(\alpha)}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I_{\xi_1^+,k}^{\alpha}\psi(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))+I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-,k}^{\alpha}\psi(\xi_1)\right]$$

$$\leq \left[\beta\left(\frac{\alpha}{k},1-s\right)+\frac{k}{\alpha-ks}\right]\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]. \tag{2.20}$$

Corollary 2.17. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\frac{k^2\Gamma_k^2(\alpha)}{2[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I_{\xi_1^+,k}^{\alpha}\psi(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))+I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-,k}^{\alpha}\psi(\xi_1)\right]$$



$$\leq \frac{k^2\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]}{(\alpha+k)(\alpha+2k)}. \tag{2.21}$$

Corollary 2.18. Under assumptions of Theorem 2.1, choosing $h_1(\imath)=\frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath)=\frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$\frac{k^2\Gamma_k^2(\alpha)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\xi_1^+,k}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi(\xi_1)\right]$$

$$\leq\left[\beta\left(\frac{\alpha}{k}+\frac{1}{2},\frac{1}{2}\right)+\beta\left(\frac{\alpha}{k}-\frac{1}{2},\frac{3}{2}\right)\right]\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right]. \tag{2.22}$$

Case 4. Taking $\phi(\imath)=\imath\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\imath\right)^{\alpha-1}$ and $\alpha\in(0,1)$, we obtain the following inequalities for the conformable fractional integral:

Corollary 2.19. Under assumptions of Theorem 2.1, taking $h_1(\imath)=h_2(\imath)=1$, we have

$$\frac{1}{[\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\alpha}-\xi_1^{\alpha}}\times\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\right]$$

$$\leq\frac{\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}}{\alpha}. \tag{2.23}$$

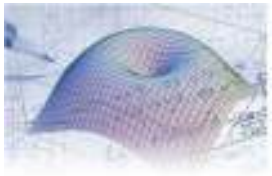
Corollary 2.20. Under assumptions of Theorem 2.1, choosing $h_1(\imath)=h(1-\imath)$ and $h_2(\imath)=h(\imath)$, we get

$$I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\leq H^{\circ}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right], \tag{2.24}$$

where

$$H^{\circ}:=\int_0^1[h(\imath)+h(1-\imath)][\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\alpha-1}d\imath.$$

Corollary 2.21. Under assumptions of Theorem 2.1, taking $h_1(\imath)=(1-\imath)^s$ and $h_2(\imath)=\imath^s$ for $s\in(0,1]$, we obtain



$$(2.25)\qquad \mathrm{I}^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+\mathrm{I}^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\le \mathrm{H}^{\circ\circ}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$\mathrm{H}^{\circ\circ}:=\frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{s+1}}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\imath^{\alpha-1}\left[(\imath-\xi_1)^{s}+\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\imath\right)^{s}\right]d\imath.$$

Corollary 2.22. Under assumptions of Theorem 2.1, choosing $\mathrm{h}_1(\imath)=(1-\imath)^{-s}$ and $\mathrm{h}_2(\imath)=\imath^{-s}$ for $s\in(0,1)$, we have

$$(2.26)\qquad \mathrm{I}^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+\mathrm{I}^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\le \mathrm{P}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$\mathrm{P}:=\frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{1-s}}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\imath^{\alpha-1}\left[\frac{1}{(\imath-\xi_1)^{s}}+\frac{1}{\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\imath\right)^{s}}\right]d\imath.$$

Corollary 2.23. Under assumptions of Theorem 2.1, taking $\mathrm{h}_1(\imath)=\mathrm{h}_2(\imath)=\imath(1-\imath)$, we get

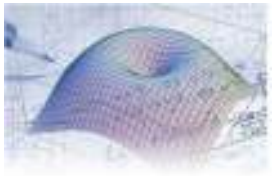
$$(2.27)\qquad \mathrm{I}^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+\mathrm{I}^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\le \mathrm{L}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$\mathrm{L}:=\frac{2}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{2}}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\imath^{\alpha-1}(\imath-\xi_1)\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\imath\right)d\imath.$$

Corollary 2.24. Under assumptions of Theorem 2.1, choosing $\mathrm{h}_1(\imath)=\frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $\mathrm{h}_2(\imath)=\frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$(2.28)\qquad \mathrm{I}^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+\mathrm{I}^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\le \mathrm{T}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$



$$T := \frac{1}{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}$$

$$\times \int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \iota^{\alpha-1}\left\{\sqrt{\frac{\iota-\xi_1}{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\iota}}+\sqrt{\frac{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)-\iota}{\iota-\xi_1}}\right\}d\iota.$$

Case 5. Taking $\phi(\iota)=\frac{\iota}{\alpha}\exp(-A\iota)$ where $A=\frac{1-\alpha}{\alpha}$ and $\alpha\in(0,1)$, we obtain the following inequalities for the fractional integral with exponential kernel:

Corollary 2.25. Under assumptions of Theorem 2.1, taking $h_1(\iota)=h_2(\iota)=1$, we have

$$\frac{(1-\alpha)}{2[1-\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)]}\times\left[I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\right]$$

$$\leq \frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}. \tag{2.29}$$

Corollary 2.26. Under assumptions of Theorem 2.1, choosing $h_1(\iota)=h(1-\iota)$ and $h_2(\iota)=h(\iota)$, we get

$$I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\leq Q\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right], \tag{2.30}$$

Where

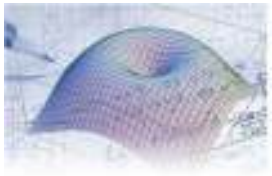
$$Q := \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}{\alpha}\int_0^1[h(\iota)+h(1-\iota)]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\iota\right)d\iota.$$

Corollary 2.27. Under assumptions of Theorem 2.1, taking $h_1(\iota)=(1-\iota)^s$ and $h_2(\iota)=\iota^s$ for $s\in(0,1]$, we obtain

$$I^{\alpha}_{\xi_1^+}\psi(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))+I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1)\leq Q^*\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}}+\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right], \tag{2.31}$$

where

$$Q^* := \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}{\alpha}\int_0^1[\iota^s+(1-\iota)^s]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\iota\right)d\iota.$$



Corollary 2.28. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$(2.32)\quad I^{\alpha}_{\xi_1^+}\psi(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) + I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1) \le Q^{**}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}} + \frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$Q^{**} := \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}{\alpha}\int_0^1 \left[\frac{1}{\imath^s} + \frac{1}{(1-\imath)^s}\right]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath\right)d\imath.$$

Corollary 2.29. Under assumptions of Theorem 2.1, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$(2.33)\quad I^{\alpha}_{\xi_1^+}\psi(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) + I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1) \le Q^{\circ}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}} + \frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$Q^{\circ} := \frac{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}{\alpha}\int_0^1 [\imath(1-\imath)]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath\right)d\imath.$$

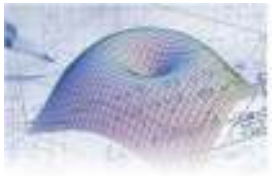
Corollary 2.30. Under assumptions of Theorem 2.1, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$(2.34)\quad I^{\alpha}_{\xi_1^+}\psi(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) + I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi(\xi_1) \le Q^{\circ\circ}\left[\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}} + \frac{\psi(\xi_2)}{e^{\omega_2\xi_2}}\right],$$

where

$$Q^{\circ\circ} := \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}{2\alpha}\int_0^1 \frac{\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath\right)}{\sqrt{\imath(1-\imath)}}d\imath.$$

Theorem 2.2. Let $h_1, h_2 : [0,1] \to [0,+\infty)$ be two continuous functions and $\psi_1, \psi_2 : \Delta \to (0,+\infty)$ are exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex and exponentially $(\varpi_1, \varpi_2, h_1, h_2)$–nonconvex functions, respectively. Then the following inequality for the generalized fractional integral holds:



$$\xi_1 + I_\phi \psi_1\left(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)\psi_2\left(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right) + {}_{\left(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)}- I_\phi \psi_1(\xi_1)\psi_2(\xi_1)$$

$$\leq C_1P_1 + C_2P_2 + C_3P_3, \tag{2.35}$$

where

$$C_1 := \frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}, \quad C_2 := \frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}, \quad C_3 := \frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}} + \frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}},$$

and

$$P_1 := \int_0^1 [\,h_1^2(\imath) + h_1^2(1-\imath)]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath)}{\imath}\,d\imath,$$

$$P_2 := \int_0^1 [\,h_2^2(\imath) + h_2^2(1-\imath)]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath)}{\imath}\,d\imath,$$

$$P_3 := \int_0^1 [\,h_1(\imath)h_2(\imath) + h_1(1-\imath)h_2(1-\imath)]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath)}{\imath}\,d\imath.$$

Proof. Since ψ_1, ψ_2 are respectively, exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex and exponentially $(\varpi_1, \varpi_2, h_1, h_2)$–nonconvex functions, we have

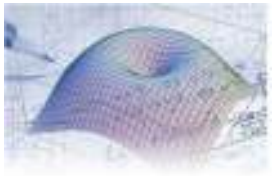
$$\psi_1\left(\xi_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)\psi_2\left(\xi_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)$$

$$\leq \left(h_1(\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}} + h_2(\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\right)\times\left(h_1(\imath)\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}} + h_2(\imath)\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}\right)$$

$$= {h_1}^2(\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}} + {h_2}^2(\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}$$

$$+\ h_1(\imath)h_2(\imath)\left(\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}} + \frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}\right) \tag{2.36}$$

and



$$\psi_1\left(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)\psi_2\left(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)$$

$$\leq\left(h_1(1-\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}+h_2(1-\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\right)\times\left(h_1(1-\imath)\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}+h_2(1-\imath)\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}\right)$$

$$=h_1{}^2(1-\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}+h_2{}^2(1-\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}$$

$$+\ h_1(1-\imath)h_2(1-\imath)\left(\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}+\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}\right). \tag{2.37}$$

Adding (2.36) and (2.37), we have

$$\psi_1\left(\xi_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)\psi_2\left(\xi_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)$$

$$+\psi_1\left(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)\psi_2\left(\xi_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\right)$$

$$\leq h_1{}^2(\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}+h_2{}^2(\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}$$

$$+h_1(\imath)h_2(\imath)\left(\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}+\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}\right)$$

$$+h_1{}^2(1-\imath)\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}+h_2{}^2(1-\imath)\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}$$

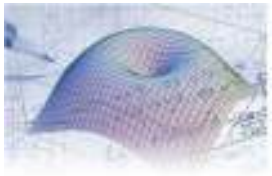
$$+\ h_1(1-\imath)h_2(1-\imath)\left(\frac{\psi_1(\xi_1)}{e^{\omega_1\xi_1}}\frac{\psi_2(\xi_2)}{e^{\varpi_2\xi_2}}+\frac{\psi_1(\xi_2)}{e^{\omega_2\xi_2}}\frac{\psi_2(\xi_1)}{e^{\varpi_1\xi_1}}\right). \tag{2.38}$$

Multiplying (2.38) with $\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)\imath)}{\imath}$ on both sides and integrating the obtained inequality with respect to $\imath$ over $[0,1]$, then we have our required inequality (2.35).

We point out some special cases of Theorem 2.2.

Case 1. Taking $\phi(\imath)=\imath$, we obtain the following inequalities for the Riemann integral:

Corollary 2.31. Under assumptions of Theorem 2.2, taking $h_1(\imath)=h_2(\imath)=1$, we have



$$(2.39)\qquad \frac{1}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^3}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \psi_1(\tau)\psi_2(\tau)d\tau \le C_1+C_2+C_3.$$

Corollary 2.32. Under assumptions of Theorem 2.2, choosing $h_1(\imath)=h(1-\imath)$ and $h_2(\imath)=h(\imath)$, we get

$$(2.40)\qquad \frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^3}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \psi_1(\tau)\psi_2(\tau)d\tau \le L_1[C_1+C_2]+C_3L_2,$$

where

$$L_1:=2\int_0^1 h^2(\imath)d\imath,\quad L_2:=2\int_0^1 h(\imath)h(1-\imath)d\imath.$$

Corollary 2.33. Under assumptions of Theorem 2.2, taking $h_1(\imath)=(1-\imath)^s$ and $h_2(\imath)=\imath^s$ for $s\in(0,1]$, we obtain

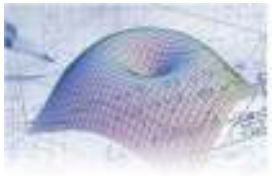
$$(2.41)\qquad \frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^3}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \psi_1(\tau)\psi_2(\tau)d\tau \le \frac{2[C_1+C_2]}{2s+1}+2C_3\beta(s+1,s+1).$$

Corollary 2.34. Under assumptions of Theorem 2.2, choosing $h_1(\imath)=(1-\imath)^{-s}$ and $h_2(\imath)=\imath^{-s}$ for $s\in(0,1)$, we have

$$(2.42)\qquad \frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^3}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \psi_1(\tau)\psi_2(\tau)d\tau \le \frac{2[C_1+C_2]}{1-2s}+2C_3\beta(1-s,1-s).$$

Corollary 2.35. Under assumptions of Theorem 2.2, taking $h_1(\imath)=h_2(\imath)=\imath(1-\imath)$, we get

$$(2.43)\qquad \frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^3}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)} \psi_1(\tau)\psi_2(\tau)d\tau \le \frac{4[C_1+C_2+C_3]}{3}.$$



Case 2. Taking $\phi(\imath) = \frac{\imath^{\alpha}}{\Gamma(\alpha)}$we obtain the following inequalities for the Riemann fractional integral:

Corollary 2.36. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{3\alpha}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$(2.44) \qquad \leq C_1 + C_2 + C_3.$$

Corollary 2.37. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{3\alpha}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$(2.45) \qquad \leq L_1[C_1 + C_2] + C_3L_2,$$

where L_1 and L_2 are defined as in Corollary 2.32.

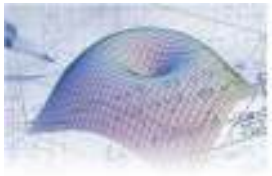
Corollary 2.38. Under assumptions of Theorem 2.2, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{3\alpha}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$(2.46) \qquad \leq \frac{2[C_1 + C_2]}{2s+1} + 2C_3\beta(s+1, s+1).$$

Corollary 2.39. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{3\alpha}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$



$$\leq \frac{2[C_1 + C_2]}{1-2s} + 2C_3\beta(1-s, 1-s). \tag{2.47}$$

Corollary 2.40. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{3\alpha}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$\leq \frac{4[C_1 + C_2 + C_3]}{3}. \tag{2.48}$$

Case 3. Taking $\phi(\imath) = \frac{\imath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we obtain the following inequalities for the k–Riemann fractional integral:

Corollary 2.41. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\xi_1^+,k}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$\leq C_1 + C_2 + C_3. \tag{2.49}$$

Corollary 2.42. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

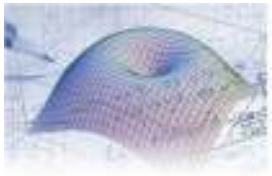
$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\xi_1^+,k}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$

$$\leq L_1[C_1 + C_2] + C_3L_2, \tag{2.50}$$

where L_1 and L_2 are defined as in Corollary 2.32.

Corollary 2.43. Under assumptions of Theorem 2.2, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\xi_1^+,k}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi_1(\xi_1)\psi_2(\xi_1)\end{bmatrix}$$



$$\leq \frac{2[C_1 + C_2]}{2s+1} + 2C_3\beta(s+1, s+1). \tag{2.51}$$

Corollary 2.44. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\xi_1^+,k}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi_1(\xi_1)\psi_2(\xi_1) \end{bmatrix}$$

$$\leq \frac{2[C_1 + C_2]}{1-2s} + 2C_3\beta(1-s, 1-s). \tag{2.52}$$

Corollary 2.45. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\xi_1^+,k}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-,k}\psi_1(\xi_1)\psi_2(\xi_1) \end{bmatrix}$$

$$\leq \frac{4[C_1 + C_2 + C_3]}{3}. \tag{2.53}$$

Case 4. Taking $\phi(\imath) = \imath\left(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1) - \imath\right)^{\alpha-1}$ and $\alpha \in (0,1)$, we obtain the following inequalities for the conformable fractional integral:

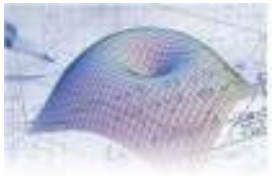
Corollary 2.46. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\frac{\alpha}{2\{[\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)]^{\alpha} - \xi_1^{\alpha}\}}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1) \end{bmatrix}$$

$$\leq C_1 + C_2 + C_3. \tag{2.54}$$

Corollary 2.47. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)}\begin{bmatrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1)) \\ +I^{\alpha}_{(\xi_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2-\xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1) \end{bmatrix}$$



$$\leq A_1[C_1 + C_2] + C_3A_2, \tag{2.55}$$

where

$$A_1 := \int_0^1 [h^2(\imath) + h^2(1-\imath)][\xi_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)]^{\alpha-1} d\imath,$$

$$A_2 := 2\int_0^1 h(\imath)h(1-\imath)[\xi_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)]^{\alpha-1} d\imath.$$

Corollary 2.48. Under assumptions of Theorem 2.2, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)}\left[\begin{matrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))\psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \\ + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1) \end{matrix}\right]$$

$$\leq A_1^{\circ}[C_1 + C_2] + C_3A_2^{\circ}, \tag{2.56}$$

Where

$$A_1^{\circ} := \frac{1}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)]^{2s+1}}\int_{\xi_1}^{\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)} \imath^{\alpha-1}\left[(1-\xi_1)^{2s} + \left(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1) - \imath\right)^{2s}\right] d\imath,$$

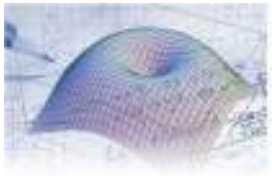
$$A_2^{\circ} := \frac{2}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)]^{2s+1}}\int_{\xi_1}^{\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)} \imath^{\alpha-1}\left[(1-\xi_1)\left(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1) - \imath\right)\right]^s d\imath.$$

Corollary 2.49. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)}\left[\begin{matrix} I^{\alpha}_{\xi_1^+}\psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))\psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \\ + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-}\psi_1(\xi_1)\psi_2(\xi_1) \end{matrix}\right]$$

$$\leq A_1^{\circ\circ}[C_1 + C_2] + C_3A_2^{\circ\circ}, \tag{2.57}$$

where



$$A_1^{\diamond\diamond} := \frac{1}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{1-2s}}\int_{\xi_1}^{\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)} \imath^{\alpha-1}\left[(\imath-\xi_1)^{-2s} + \left(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)-\imath\right)^{-2s}\right]d\imath,$$

$$A_2^{\diamond\diamond} := \frac{2}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^{1-2s}}\int_{\xi_1}^{\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)} \imath^{\alpha-1}\left[(\imath-\xi_1)\left(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)-\imath\right)\right]^{-s}d\imath.$$

Corollary 2.50. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

$$\left[I_{\xi_1^+}^{\alpha}\psi_1(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)) + I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-}^{\alpha}\psi_1(\xi_1)\psi_2(\xi_1)\right]$$

$$(2.58) \qquad \leq A^*[C_1+C_2+C_3],$$

where

$$A^* := \frac{2}{[\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)]^4}\int_{\xi_1}^{\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)} \imath^{\alpha-1}\left[(\imath-\xi_1)\left(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)-\imath\right)\right]^2 d\imath.$$

Case 5. Taking $\phi(\imath) = \frac{\imath}{\alpha}\exp(-A\imath)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$, we obtain the following inequalities for the fractional integral with exponential kernel:

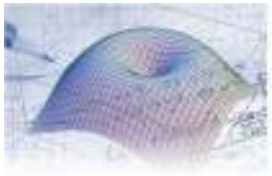
Corollary 2.51. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = 1$, we have

$$\frac{(1-\alpha)}{2[1-\exp\left(-A\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)\right)]} \times \left[I_{\xi_1^+}^{\alpha}\psi_1(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))\psi_2(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)) + I_{(\xi_1+\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1))^-}^{\alpha}\psi_1(\xi_1)\psi_2(\xi_1)\right]$$

$$(2.59) \qquad \leq C_1+C_2+C_3.$$

Corollary 2.52. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get

$$\frac{\alpha}{\mathcal{F}_{\rho,\lambda}^{\sigma}(\xi_2-\xi_1)}$$



$$\times \left[I^{\alpha}_{\xi_1^+} \psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-} \psi_1(\xi_1)\psi_2(\xi_1) \right]$$

$$(2.60) \qquad \leq \overline{P}[C_1 + C_2] + C_3\overline{R},$$

where

$$\overline{P} := \int_0^1 [h^2(\imath) + h^2(1-\imath)] \exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right) d\imath,$$

$$\overline{R} := 2\int_0^1 [h(\imath)h(1-\imath)] \exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right) d\imath.$$

Corollary 2.53. Under assumptions of Theorem 2.2, taking $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$ for $s \in (0,1]$, we obtain

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)}$$

$$\times \left[I^{\alpha}_{\xi_1^+} \psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-} \psi_1(\xi_1)\psi_2(\xi_1) \right]$$

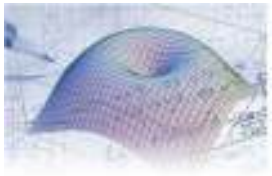
$$(2.61) \qquad \leq \overline{P}_1[C_1 + C_2] + C_3\overline{R}_1,$$

Where $\overline{P}_1 := \int_0^1 [\imath^{2s} + (1-\imath)^{2s}] \exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right) d\imath,$

$$\overline{R}_1 := 2\int_0^1 [\imath(1-\imath)]^s \exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right) d\imath.$$

Corollary 2.54. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$ for $s \in (0,1)$, we have

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)}$$



$$\times \Big[I^{\alpha}_{\xi_1^+} \psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-} \psi_1(\xi_1) \psi_2(\xi_1) \Big]$$

$$\leq \overline{P}_2 [C_1 + C_2] + C_3 \overline{R}_2, \tag{2.62}$$

where

$$\overline{P}_2 := \int_0^1 [\imath^{-2s} + (1-\imath)^{-2s}] \exp\big(-A \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\big) d\imath,$$

$$\overline{R}_2 := 2 \int_0^1 [\imath(1-\imath)]^{-s} \exp\big(-A \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\big) d\imath.$$

Corollary 2.55. Under assumptions of Theorem 2.2, taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get

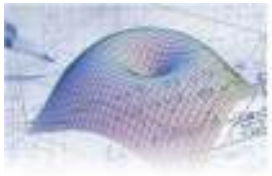
$$\frac{\alpha}{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)} \times \Big[I^{\alpha}_{\xi_1^+} \psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-} \psi_1(\xi_1) \psi_2(\xi_1) \Big]$$

$$\leq \overline{P}_3 [C_1 + C_2 + C_3], \tag{2.63}$$

Where $\overline{P}_3 := \int_0^1 [\imath(1-\imath)]^2 \exp\big(-A \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\big) d\imath.$

Corollary 2.56. Under assumptions of Theorem 2.2, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we obtain

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)} \times \Big[I^{\alpha}_{\xi_1^+} \psi_1(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) \psi_2(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)) + I^{\alpha}_{(\xi_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1))^-} \psi_1(\xi_1) \psi_2(\xi_1) \Big]$$



$$(2.64) \quad \leq \overline{P}_4[C_1 + C_2] + \overline{R}_4 C_3,$$

where

$$\overline{P}_4 := \int_0^1 [\frac{\imath}{1-\imath} + \frac{1-\imath}{\imath}]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right)d\imath,$$

$$\overline{R}_4 := \frac{\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)\imath\right)}{2A\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1)}.$$

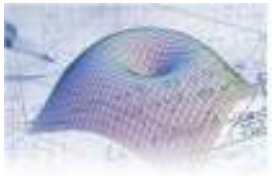
Remark 2.1. If we take $\omega_i = \varpi_i = 0$ for all $i = 1,2$ in all proved results of this paper, then all results holds for the (h_1, h_2)–nonconvex functions.

Remark 2.2. If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\xi_2 - \xi_1) = \xi_2 - \xi_1$ in our Theorems 2.1 and 2.2, then all results holds for the exponentially $(\omega_1, \omega_2, h_1, h_2)$–convex functions.

Remark 2.3. For different positive values of ρ, λ, where $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ is bounded sequence of positive real numbers involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function) in our Theorems 2.1 and 2.2, we have different fascinating inequalities of Hermite–Hadamard type. Moreover, from our corollaries some applications to special means for different positive real numbers and bounds regarding error estimations for quadrature formula can be found as well. This demonstrates the efficient of our results. We omit here their proofs and the details are left to the interested reader.

3. Conclusion

In this paper, the authors defined new class of functions, the so-called exponentially $(\omega_1, \omega_2, h_1, h_2)$–nonconvex function involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function). Utilizing this, some Hermite–Hadamard type integral inequalities via generalized fractional integral operator are provided. Under suitable condition regarding parameters ω_1 and ω_2, the results hold at the same time for all convex functions. Interested reader can establish new inequalities via fractional operators or multiplicative integrals. We believe that our results can be applied in convex analysis, quantum calculus, optimization and different areas of pure and applied sciences.



References

[1] Tomar, M., Agarwal, P., Jleli, M., Samet, B.: Certain Ostrowski type inequalities for generalized s–convex functions. J. Nonlinear Sci. Appl. **10**, 5947–5957 (2017).

[2] Varoŝanec, S.: On h–convexity. J. Math. Anal. Appl. **326**, 303–311 (2007).

[3] Awan, M.U., Noor, M.A., Noor, K.I.: Hermite–Hadamard inequalities for exponentially convex functions. Appl. Math. Inf. Sci. **12**, 405–409 (2018).

[4] Tomar, M., Mubeen, S., Choi, J.: Certain inequalities associated with Hadamard k-fractional integral operators. J. Inequal. Appl. **2016**(234), (2016).

[5] Baleanu, D., Kashuri, A., Mohammed, P.O., Meftah, B.: General Raina fractional integral inequalities on coordinates of convex functions. Adv. Difference Equ. **2021**(82), (2021).

[6] Mohammed, P.O., Abdeljawad, T., Zeng, S., Kashuri, A.: Fractional Hermite–Hadamard integral inequalities for a new class of convex functions. Symmetry **12**, 1485, (2020).

[7] Kashuri, A., Liko, R.: Some new Hermite–Hadamard type inequalities and their applications. Stud. Sci. Math. Hung. **56**(1), 103–142 (2019).

[8] Kashuri, A., Iqbal, S., Butt, S.I., Nasir, J., Nisar, K.S., Abdeljawad, T.: Trapezium-type inequalities for k–fractional integral via new exponential-type convexity and their applications. J. Math. **2020**, (2020), Article ID 8672710.

[9] Awan, M.U., Noor, M.A., Noor, K.I., Safdar, F.: On strongly generalized convex functions. Filomat **31**(18), 5783–5790 (2017).

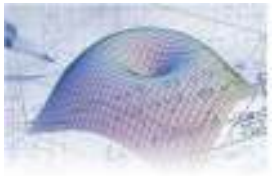
[10] Delavar, M.R., De La Sen, M.: Some generalizations of Hermite–Hadamard type inequalities. SpringerPlus **5**, (2016).

[11] Gordji, M.E., Delavar, M.R., De La Sen, M.: On ϕ–convex functions. J. Math. Inequal. **10**(1), 173–183 (2016).

[12] Jleli, M., Regan, D.O., Samet, B.: On Hermite–Hadamard type inequalities via generalized fractional integrals. Turk. J. Math. **40**, 1221–1230 (2016).

[13] Set, E., Tomar, M., Sarikaya, M.Z.: On generalized Grüss type inequalities via k–Riemann–Liouville fractional integral. Appl. Math. Comput. **269**, 29–34 (2015).

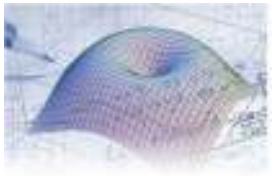
[14] Saddiqa, M., Farid, G., Ullah, S., Jung, C.Y., Shim, S.H.: On bounds of fractional integral operators containing Mittag–Leffler functions for generalized exponentially convex functions. AIMS Math. **6**(6), 6454–6468 (2021).



[15] Raina, R.K.: On generalized Wright's hypergeometric functions and fractional calculus operators. East Asian Math. J. **21**(2), 191–203 (2005).

[16] Sarikaya, M.Z., Ertuğral, F.: On the generalized Hermite–Hadamard inequalities. (2017).

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



On the Rough Statistical Convergence of a Sequence of Sets

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Abstract: In this paper, we give the equivalent definitions for both rough statistical Wijsman convergence and rough statistical Hausdorff convergence of a sequence of sets. Then we prove the relation between rough statistical Wijsman convergence and rough statistical Hausdorff convergence.

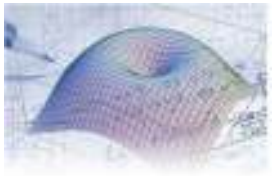
Keywords: Sequence of sets; Rough statistical Wijsman convergence; Rough statistical Hausdorff convergence.

1. Introduction

The notion of statistical convergence of sequences was introduced by Fast [4] and Schoenberg independently [14]. Fridy [5] defined the concept of a statistically Cauchy sequence and prove that it is equivalent to statistical convergence. Fridy and Orhan [6] gave the definitions of statistical limit superior and statistical limit inferior.

Wijsman [17,18] defined a new concept of convergence by using the pointwise convergence of sequences obtained from distance functions. If this pointwise convergence is replaced to uniform convergence, then Hausdorff convergence is obtained. Nuray and Rhoades [7] generalized the concepts of Wijsman convergence and Hausdorff convergence using the theory of statistical convergence. Later, Nuray et al. [8] gave the notion of Wijsman statistical convergence of double sequences of sets.

The rough convergence theory was first introduced by Phu [12] in 2001. Aytar [1] gave a new definition by combining rough convergence theory with the statistical convergence theory. Debnath and Rakshit [2] studied on rough convergence in metric spaces.



Recently, the rough convergence theory has started to be applied in set theory as well. Ölmez and Aytar [9] first defined the notion of rough Wijsman convergence. Ölmez et al. [10] stated the equivalent definition of rough Wijsman convergence. Subramanian and Esi [15,16] introduced the concepts of rough Wijsman convergence and rough Wijsman statistical convergence for a triple sequence of sets, respectively.

In this paper, the equivalent definitions of rough Wijsman statistical convergence and of rough Hausdorff statistical convergence are given. Then the relation between rough Wijsman convergence and rough Hausdorff convergence is examined (see Theorem 3.1). The proof techniques are similar to that of classical case (see [10]).

2. Preliminaries

Throughout this paper, we assume that X is a nonempty set and ρ_X is a metric on X and A, A_n are nonempty closed subsets of X for each $n \in \mathbb{N}$.

We briefly recall some of the basic notions in the theory of statistical convergence and we refer readers to [4,5,6,14] for more details. Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \le n\}$ by K_n. Then the natural density of K is given by $\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n.

The sequence $\{x_n\}$ is said to be *statistically convergent* to x, denoted by $x_n \xrightarrow{st} x$ or $st - \lim x_n = x$, if for each $\varepsilon > 0$ the set

$$\{n \in \mathbb{N} \ : \ \rho_X(x_n, x) \ge \varepsilon\}$$

has natural density zero [5].

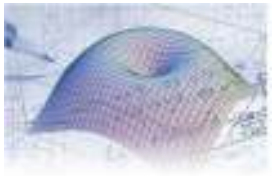
For a sequence $\{x_n\}$, the notions of *statistical limit superior* and *statistical limit inferior* are defined as follows:

$$st - \limsup\{x_n\} := \begin{cases} \sup B_{\{x_n\}} & , B_{\{x_n\}} \neq \varnothing \\ -\infty & , \text{otherwise} \end{cases},$$

$$st - \liminf\{x_n\} := \begin{cases} \inf A_{\{x_n\}} & , A_{\{x_n\}} \neq \varnothing \\ +\infty & , \text{otherwise} \end{cases},$$

where

$$A_{\{x_n\}} := \{a \in \mathbb{R} \ : \ \delta(\{n \in \mathbb{N} \ : \ x_n < a\}) \neq 0\}$$



and

$$B_{\{x_n\}} := \{b \in \mathsf{R} : \delta(\{n \in \mathsf{N} : x_n > b\}) \neq 0\} \text{ [6].}$$

Let r be a nonnegative real number. The sequence $\{x_n\}$ is said to be *rough convergent* to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathsf{N}$ such that $\rho_X(x_n, x) < r + \varepsilon$ for each $n \geq n(\varepsilon)$ [12].
The *distance function* $d(\cdot, A): X \to [0, \infty)$ is defined by the formula

$$d(x, A) = \inf\{\rho_X(x, y): y \in A\} \text{ [17].}$$

We say that the sequence $\{A_n\}$ is *Wijsman convergent* to the set A if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \xrightarrow{W} A$, as $n \to \infty$ [17].
We say that the sequence $\{A_n\}$ is *Wijsman statistically convergent* to the set A if

$$\delta(\{n \in \mathsf{N} : |d(x, A_n) - d(x, A)| \geq \varepsilon\}) = 0$$

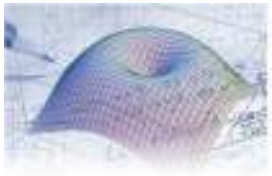
for every $\varepsilon > 0$ and each $x \in X$. In this case, we write $A_n \xrightarrow{st-W} A$, as $n \to \infty$ [7].

3. Rough Statistical Convergence

In this study, inspired by the work of Ölmez et al. [10], we obtained some results for rough statistical Wijsman convergence and rough Hausdorff statistical convergence.
Definition 3.1 ([9]). A sequence $\{A_n\}$ is said to be $r-$*Wijsman convergent* to the set A if for every $\varepsilon > 0$ and each $x \in X$ there is an $N(\varepsilon, x) \in \mathsf{N}$ such that

$$|d(x, A_n) - d(x, A)| < r + \varepsilon \text{ for all } n \geq N(\varepsilon, x)$$

and we write $d(x, A_n) \xrightarrow{r} d(x, A)$ or $A_n \xrightarrow{r-W} A$ as $n \to \infty$.
Now define



(3.1) $$1-LIM^{r}A_n=\{A\subset X\ :\ A_n\overset{r-W}{\to}A\}.$$

Definition 3.2 ([16]). A sequence $\{A_n\}$ is said to be $r-$*Wijsman statistically convergent* to the set A if we have

$$\delta\left(\{n\in\mathbb{N}\ :\ |d(x,A_n)-d(x,A)|\geq r+\varepsilon\}\right)=0$$

for every $\varepsilon>0$ and each $x\in X$ and we write $d(x,A_n)\overset{r-st}{\to}d(x,A)$ or $A_n\overset{r-st-W}{\to}A$ as $n\to\infty$.

Now define

(3.2) $$st_1-LIM^{r}A_n=\{A\subset X\ :\ A_n\overset{r-st-W}{\to}A\}.$$

If a sequence is Wijsman statistically convergent, then this sequence is $r-$Wijsman statistically convergent to the same set for each r.

The following proposition characterizes $r-$Wijsman statistical convergence by means of the upper limit.

Proposition 3.1. The set

$$\left\{n\in\mathbb{N}\ :\ |d(x,A_n)-d(x,A)|\geq r+\varepsilon\right\}$$

has natural density zero for every $\varepsilon>0$ and each $x\in X$ if and only if

$$st-\limsup_{n\to\infty}|d(x,A_n)-d(x,A)|\leq r.$$

Proof. *(Necessity)* Assume that $d(x,A_n)\overset{r-st}{\to}d(x,A)$. Define

$$f_n(x):=|d(x,A_n)-d(x,A)|.$$

Then for every $\varepsilon>0$ and each $x\in X$ the set $\left\{n\in\mathbb{N}\ :\ f_n(x)\geq r+\varepsilon\right\}$ has natural density zero. On the contrary, suppose that

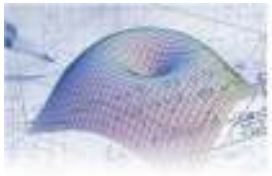
$$st-\limsup_{n\to\infty}f_n(x)>r.$$

Let

$$\delta(x)=\frac{st-\limsup_{n\to\infty}f_n(x)-r}{3}.$$

By definition of $st-\limsup$, we have

$$f_n(x)>st-\limsup_{n\to\infty}f_n(x)-\delta(x)$$



for each $n \in K$, where K is a subset of $\mathbb{N}$ with $\delta(K) \neq 0$. Take $\varepsilon = \eta(x)$, then we have

(3.3) $$\delta\left(\left\{n \in \mathbb{N} : f_n(x) \geq r + \eta(x)\right\}\right) = 0.$$

The expression (3.3) contradicts to the definition of $st - \limsup$. Hence, we get $st - \limsup_{n\to\infty} f_n(x) \leq r$ for each $x \in X$.

(Sufficiency) Now let us $st - \limsup_{n\to\infty} f_n(x) \leq r$ for each $x \in X$. On the contrary, suppose that there exists an $\tilde{\varepsilon} > 0$ and an $x \in X$ such that

$$\delta\left(\left\{n \in \mathbb{N} : f_n(x) < r + \tilde{\varepsilon}\right\}\right) \neq 0.$$

By the assumption, we have

(3.4) $$\delta\left(\left\{n \in \mathbb{N} : f_n(x) > \left[st - \limsup_{n\to\infty} f_n(x)\right] + \tilde{\varepsilon}\right\}\right) \neq 0.$$

This expression (3.4) contradicts to the fact that

$$\delta\left(\left\{n \in \mathbb{N} : f_n(x) < r + \tilde{\varepsilon}\right\}\right) \neq 0,$$

which completes the proof.

We say that a sequence $\{A_n\}$ is *Hausdorff convergent* to the set A if $\lim_{n\to\infty} H(A_n, A) = 0$, and we denote $A_n \overset{H}{\to} A$, as $n \to \infty$. That is, we have $A_n \overset{H}{\to} A$ provided that, for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H(A_n, A) = \sup_{x\in X} \left|d(x, A_n) - d(x, A)\right| < \varepsilon \text{ for all } n \geq N(\varepsilon) \text{ [13].}$$

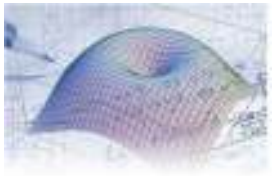
The sequence $\{A_n\}$ is said to be $r-$*Hausdorff convergent* to the set A if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H(A_n, A) = \sup_{x\in X} \left|d(x, A_n) - d(x, A)\right| < r + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $A_n \overset{r-H}{\to} A$ as $n \to \infty$ [10].We say that a sequence $\{A_n\}$ is *Hausdorff statistically convergent* to the set A if $st - \lim_{n\to\infty} H(A_n, A) = 0$, and we denote $A_n \overset{st-H}{\to} A$, as $n \to \infty$. That is, we have $A_n \overset{st-H}{\to} A$ provided that, the set

$$\left\{n \in \mathbb{N} : \sup_{x\in X} \left|d(x, A_n) - d(x, A)\right| \geq \varepsilon\right\}$$

has natural density zero for every $\varepsilon > 0$ [7].



Example 3.1. Define a sequence $\{A_n\}$ by

$$A_n := \begin{cases} \left[-3,-1-\frac{1}{n}\right]\times[-1,1] & \text{, if } n \text{ is a nonsquare number} \\ [-1,n]\times[-1,1] & \text{, otherwise} \end{cases}.$$

This sequence is Hausdorff statistically convergent to the set $A=[-3,-1]\times[-1,1]$.

The concept of rough statistical convergence was defined by Dündar and Çakan [3] and Pal et al. [11] in an infinite dimensional normed space. This definition can be easily generalized in any metric space. Moreover, since Hausdorff distance is a metric, we can define a new type of Hausdorff statistical convergence for a sequence of closed sets as follows:

Definition 3.3. The sequence $\{A_n\}$ is said to be $r-$*Hausdorff statistically convergent* to the set A if

$$\delta\left(\left\{n\in\mathbb{N} \ : \ \sup_{x\in X}\left|d(x,A_n)-d(x,A)\right|\geq r+\varepsilon\right\}\right)=0$$

for every $\varepsilon>0$. In this case, we write $A_n \overset{r-st-H}{\to} A$ as $n\to\infty$.

Now let us define

$$st_H-LIM^r A_n=\left\{A\subset X \ : \ A_n \overset{r-st-H}{\to} A\right\}.$$

If a sequence is Hausdorff statistically convergent, then this sequence is $r-$Hausdorff statistically convergent to the same set for each r. However, the converse of this claim does not hold in general, as can be seen following

Example 3.2. Let us consider the sequence $\{A_n\}$ defined in Example 3.1. This sequence is $r-$Hausdorff statistically convergent to the set $A=[-3,3]\times[-1,1]$ for each $r\geq 4$.

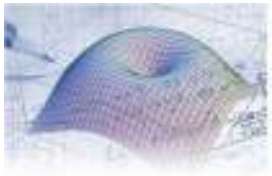
The following proposition characterizes the concept of $r-$Hausdorff statistical convergence by means of the statistical upper limit. Its proof is similar to that of Proposition 3.1.

Proposition 3.2. For every $\varepsilon>0$ we have

$$\delta\left(\left\{n\in\mathbb{N} \ : \ \sup_{x\in X}\left|d(x,A_n)-d(x,A)\right|\geq r+\varepsilon\right\}\right)=0$$

if and only if

$$st-\limsup_{n\to\infty}\sup_{x\in X}\left|d(x,A_n)-d(x,A)\right|\leq r.$$



Finally, giving the relation between the concepts of rough Wijsman statistical convergence and rough Hausdorff statistical convergence, we will end our work.

Theorem 3.1. If the sequence $\{A_n\}$ is $r-$Hausdorff statistical convergent to the set A, then it is $r-$Wijsman statistical convergent to the same set.

Proof. Let us the sequence $\{A_n\}$ is $r-$Hausdorff statistically convergent to the set A. Then, we have

$$\delta\left(\left\{n \in \mathbb{N} \ : \ \sup_{x\in X}\left|d(x,A_n)-d(x,A)\right| \geq r+\varepsilon\right\}\right)=0.$$

Thus we get

$$\left|d(x_0,A_n)-d(x_0,A)\right| < \sup_{x\in X}\left|d(x,A_n)-d(x,A)\right| < r+\varepsilon$$

for almost all n. Since $x_0 \in X$ is an arbitrary point, we say that the sequence $\{A_n\}$ is $r-$Wijsman statistically convergent to the set A.

The converse of the above theorem does not hold in general as can be seen in the following

Example 3.3. Define the sequence $\{A_n\}$ as follows:

$$A_n := \begin{cases} \{(x,y) \in \mathrm{R}^2 \ : \ x^2+y^2+(n+2)2y+4(n+1)=0\} & \text{, if } n \text{ is a nonsquare number} \cdot \\ \{(0,-n)\} & \text{, otherwise} \end{cases}$$

This sequence is $r-$Wijsman statistically convergent to the set

$$A=\{(x,y) \in \mathrm{R}^2 \ : \ x \in \mathrm{R} \text{ and } y=-1\}$$

for $r=1,$ but it is not $r-$Hausdorff statistically convergent to the set A.

References:

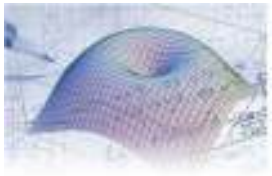
[1] Aytar, S.: Rough statistical convergence, Numer. Funct. Anal. and Optimiz. 29(3-4), 291-303 (2008).

[2] Debnath, S., Rakshit, D.: Rough convergence in metric spaces, New Trends in Analysis and Interdisciplinary Applications 449-454 (2017).

[3] Dündar, E., Çakan, C.: Rough I -Convergence, Demonstratio Mathematica 47(3), 638-651 (2014).

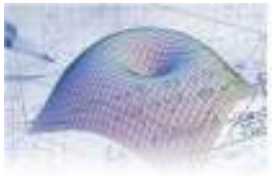
[4] Fast, H.: Sur la convergence statistique, Colloq. Math. 2, 241-244 (1951).

[5] Fridy, J.A.: On statistical convergence, Analysis 5, 301-313 (1985).



[6] Fridy, J.A., Orhan, C.: Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125(12), 3625-3631 (1997).
[7] Nuray, F., Rhoades, B.E.: Statistical convergence of sequences of sets, Fasciculi Mathematici 49, 87-99 (2012).
[8] Nuray, F., Dündar, E., Ulusu, U.: Wijsman Statistical Convergence of Double Sequences of Sets, Iranian Journal of Mathematical Sciences and Informatics 16(1), 55-64 (2021).
[9] Ölmez, Ö., Aytar, S.: The relation between rough Wijsman convergence and asymptotic cones, Turkish Journal of Mathematics 40, 1349-1355 (2016).
[10] Ölmez, Ö., Gecit Akçay, F., Aytar, S.: On the rough convergence of a sequence of sets, Electronic Journal of Mathematical Analysis and Applications 10(1), 167-174 (2022).
[11] Pal, S.K., Chandra, D., Dutta, S.: Rough ideal convergence, Hacettepe Journal of Mathematics and Statistics 42(6), 633-640 (2013).
[12] Phu, H.X.: Rough convergence in normed linear spaces, Numer. Funct. Anal. and Optimiz. 22, 201-224 (2001).
[13] Rockafellar, R.T., Wets, Roger J-B.: Variational Analysis. Springer, Berlin (1997).
[14] Schoenberg, I.J.: The integrability of certain functions and related summability methods, Amer. Math. Monthly 66, 361-375 (1959).
[15] Subramanian, N., Esi, A.: Wijsman rough convergence of triple sequences, Mat. Stud. 48, 171-179 (2017).
[16] Subramanian, N., Esi, A.: Wijsman rough lacunary statistical convergence on I Cesàro triple sequences, Int. J. Anal. Appl. 16(5), 643-653 (2018).
[17] Wijsman, R.A.: Convergence of sequences of convex sets, cones and functions, Bulletin of the American Mathematical Society 70(1), 186-188 (1964).
[18] Wijsman, R.A.: Convergence of sequences of convex sets, cones and functions II, Trans. Amer. Math. Soc. 123, 32-45 (1966).

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



Some Inclusion Relations for the Rough Statistical Wijsman Limit Set

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Abstract: In this paper, we define the set of rough statistical Wijsman limit of a sequence of sets. Then we prove some inclusion relations related to this limit set.

Keywords: Rough statistical convergence; Wijsman convergence; Rough limit set.

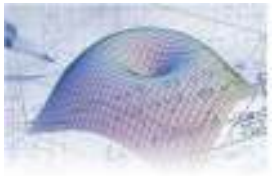
1. Introduction

The idea of statistical convergence, it was introduced by Fast [4] and Schoenberg [12] independently. Fridy [5] gave the definition of statistically Cauchy sequence. Fridy and Orhan [6] defined the notions of statistical limit superior and statistical limit inferior.

Wijsman [15,16] introduced the famous concept of Wijsman convergence, which corresponds to the pointwise convergence of sequences made up of distance functions. Nuray and Rhoades [7] extended the notion of Wijsman convergence to statistical convergence. Later, Nuray et al. [8] generalized the concept of Wijsman statistical convergence to double sequences of sets.

As for the concept of rough convergence was first introduced by Phu [11] in finite dimensional normed spaces. Aytar [2] applied the statistical convergence theory to the rough convergence theory. Debnath and Rakshit [3] gave some results related to the rough limit set in metric spaces.

Recently, many authors have studied on the theory of rough convergence of sequences of sets. Ölmez and Aytar [9] first introduced the concept of rough Wijsman convergence. Ölmez et al. [10] gave the equivalent definition of rough Wijsman



convergence. Subramanian and Esi [13,14] defined the concepts of rough Wijsman convergence and rough Wijsman statistical convergence for a triple sequence of sets, respectively.

In this paper, defining the concept of the set of rough Wijsman statistical limit, some inclusion relations related to this limit set are proved. The proof techniques are similar to that of classical case (see [10]).

2. Preliminaries

Throughout this paper, we assume that X is a nonempty set and ρ_X is a metric on X and A, A_n are nonempty closed subsets of X for each $n \in \mathbb{N}$.

We briefly recall some of the basic notions in the theory of statistical convergence and we refer readers to [4,5,6,14] for more details. Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \le n\}$ by K_n. Then the natural density of K is given by $\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n.

The sequence $\{x_n\}$ is said to be *statistically convergent* to x, denoted by $x_n \xrightarrow{st} x$ or $st - \lim x_n = x$, if for each $\varepsilon > 0$ the set

$$\{n \in \mathbb{N} : \rho_X(x_n, x) \ge \varepsilon\}$$

has natural density zero [5].

Let r be a nonnegative real number. The sequence $\{x_n\}$ is said to be *rough convergent* to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $\rho_X(x_n, x) < r + \varepsilon$ for each $n \ge n(\varepsilon)$ [11].

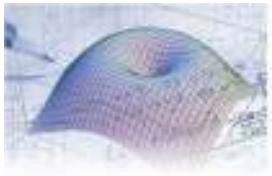
The *distance function* $d(\cdot, A) : X \to [0, \infty)$ is defined by the formula

$$d(x, A) = \inf\{\rho_X(x, y) : y \in A\}.$$

We say that the sequence $\{A_n\}$ is *Wijsman convergent* to the set A if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \xrightarrow{W} A$, as $n \to \infty$ [15].



The set A is called *Wijsman cluster point* of the sequence $\{A_n\}$ provided that there is a subsequence that Wijsman converges to the set A. In this case $L_{\{A_n\}}$ denotes the set of all cluster points of the sequence $\{A_n\}$.

We say that the sequence $\{A_n\}$ is *Wijsman statistically convergent* to the set A if

$$\delta\left(\{n \in \mathbb{N} : \left|d(x,A_n) - d(x,A)\right| \geq \varepsilon\}\right) = 0$$

for each $x \in X$ and every $\varepsilon > 0$. In this case, we write $A_n \overset{st-W}{\to} A$, as $n \to \infty$ [7].

The set A is called *Wijsman statistical cluster point* of the sequence $\{A_n\}$ provided that

$$\delta\left(\{n \in \mathbb{N} : \left|d(x,A_n) - d(x,A)\right| < \varepsilon\}\right) \neq 0$$

for each $x \in X$ and every $\varepsilon > 0$. In this case $\Gamma_{\{A_n\}}$ denotes the set of all statistical cluster points of the sequence $\{A_n\}$ [7]·

3. Rough Statistical Wijsman Limit Sets

Definition 3.1 ([9]). A sequence $\{A_n\}$ is said to be $r-$*Wijsman convergent* to the set A if for every $\varepsilon > 0$ and each $x \in X$ there is an $N(\varepsilon, x) \in \mathbb{N}$ such that

$$\left|d(x,A_n) - d(x,A)\right| < r + \varepsilon \text{ for all } n \geq N(\varepsilon,x)$$

and we write $d(x,A_n) \overset{r}{\to} d(x,A)$ or $A_n \overset{r-W}{\to} A$ as $n \to \infty$.

Definition 3.2 ([14]). A sequence $\{A_n\}$ is said to be $r-$statistically *Wijsman convergent* to the set A if we have

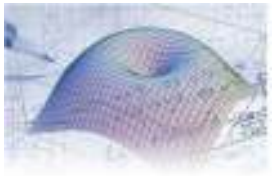
$$\delta\left(\{n \in \mathbb{N} : \left|d(x,A_n) - d(x,A)\right| \geq r + \varepsilon\}\right) = 0$$

for every $\varepsilon > 0$ and each $x \in X$ and we write $d(x,A_n) \overset{r-st}{\to} d(x,A)$ or $A_n \overset{r-st-W}{\to} A$ as $n \to \infty$.

Now define

$$st_1 - LIM^r A_n = \{A \subset X : A_n \overset{r-st-W}{\to} A\}. \tag{3.2}$$

If a sequence is Wijsman statistical convergent, then this sequence $r-$Wijsman statistical converges to the same set for each r. However, there are some sequences of



sets which are $r-$Wijsman statistically convergent, but not Wijsman statistically convergent as can be seen in the following

Example 3.1. Define a sequence $\{A_n\}$ by

$$A_n := \begin{cases} \left\{\frac{1}{n}\right\} \times [-1,0] & , \text{ if } n \text{ is an odd and nonsquare integer} \\ \left\{\frac{1}{n}\right\} \times [0,1] & , \text{ if } n \text{ is an even and nonsquare integer} \\ \{n\} \times [0,1] & , \text{ otherwise} \end{cases}.$$

We modified this sequence for the statistical convergence. You can see in [1] for the classical version of this sequence. This sequence is not statistically Wijsman convergent to the set $A = \{(0,0)\}$. Indeed, if we take $x = (0,1)$ then we have $d((0,1), A) = 1$ and

$$d((0,1), A_n) = \begin{cases} \sqrt{1+\frac{1}{n^2}} & , \text{ if } n \text{ is an odd and nonsquare integer} \\ \frac{1}{n} & , \text{ if } n \text{ is an even and nonsquare integer.} \\ n & , \text{otherwise} \end{cases}$$

Since

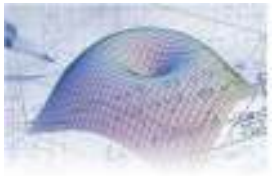
$$st - \limsup_{n\to\infty} d((0,1), A_n) = 1 \text{ and } st - \liminf_{n\to\infty} d((0,1), A_n) = 0,$$

we have that $st - \lim_{n\to\infty} d(x, A_n)$ does not exist. Therefore, this sequence is not Wijsman statistically convergent to the set A, but this sequence is $r-$Wijsman statistically convergent to the set A for $r \geq 1$.

Remark 3.1. In the Example 3.1, if we take $A \in st_1 - LIM^r A_n$, then we have

$$A \subseteq \overline{B}((0,0), r) \tag{3.3}$$

where $\overline{B}((0,0), r) = \{x \in \mathrm{R}^2 : \rho((0,0), x) \leq r\}$. Now let us prove the inclusion (3.3) . On the contrary, assume that $A \not\subseteq \overline{B}((0,0), r)$. For the sake of generality let us choose $x = (x_1, x_2) \in \mathrm{R}^2$ such that $x_1 > 0$ and $x_2 > 1$. Then, there exists an $x = (x_1, x_2) \in A$ such that $\sqrt{x_1^2 + x_2^2} > r$. Then we have



$$d(x,A_n)=\begin{cases}\sqrt{\left(x_1-\frac{1}{n}\right)^2+x_2^2} & \text{, if } n \text{ is an odd and nonsquare integer}\\ \sqrt{\left(x_1-\frac{1}{n}\right)^2+\left(x_2-1\right)^2} & \text{, if } n \text{ is an even and nonsquare integer}\\ \sqrt{\left(x_1-n\right)^2+\left(x_2-1\right)^2} & \text{, otherwise}\end{cases}.$$

Since $\left|d(x,A_n)-d(x,A)\right|>r$ with odd and nonsquare integer, we get $A\notin st_1-LIM^r A_n$. This contradiction completes the proof. If the origin is an element of the set A and A is contained in the $r-$closed ball, then we have $A\in st_1-LIM^r A_n$.

Remark 3.2. In the Example 3.1, if we take $A=\left\{\left(\frac{1}{5},0\right)\right\}$ then the sequence $\{A_n\}$ is not $r-$Wijsman statistically convergent to the set A for $r=1$. Since $r=\sup_{c\in\Gamma_{\{A_n\}}}\left\|\left(\frac{1}{5},0\right)-c\right\|=\frac{\sqrt{26}}{5}$, the sequence $\{A_n\}$ is $r-$Wijsman statistically convergent to the set $A=\left\{\left(\frac{1}{5},0\right)\right\}$ for $r=\frac{\sqrt{26}}{5}$. Moreover this sequence is not $r-$Wijsman statistically convergent to the set A for all $r<\frac{\sqrt{26}}{5}$.

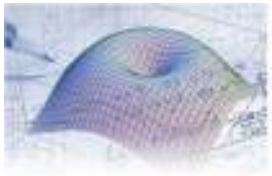
Now define

$$st_2-LIM^r A_n=\bigcap_{c\in\Gamma_{\{A_n\}}}\overline{B}(c,r)=\{A\in X\ :\ \Gamma_{\{A_n\}}\subseteq\overline{B}(A,r)\}. \tag{3.4}$$

Then, for the usual rough statistical convergence of a sequence of elements in any metric space, the sets defined by the equalities (3.2) and (3.4) coincide with each other. But these sets are not equal for an arbitrary sequence of sets. The following example compares the sets $st_1-LIM^r A_n$ and $st_2-LIM^r A_n$.

Example 3.2. Define a sequence $\{A_n\}$ by

$$A_n:=\begin{cases}\left[\frac{1}{n},3\right]\times[0,2] & \text{, if } n \text{ is an odd and nonsquare integer}\\ \left[-3,-\frac{1}{n}\right]\times[0,2] & \text{, if } n \text{ is an even and nonsquare integer.}\\ [0,n]\times[-2,0] & \text{, otherwise}\end{cases}$$



This sequence is not Wijsman statistically convergent to the set $A = \{(0,0)\} \times [0,2]$. But, this sequence is $r-$Wijsman statistically convergent to the set A for $r \geq 3$. Moreover, we have $A \in st_1 - LIM^r A_n$ and

$$\begin{aligned} st_2 - LIM^r A_n &= \overline{B}([0,3] \times [0,2],3) \cap \overline{B}([-3,0] \times [0,2],3) \\ &= [-3,3] \times [-3,5] \end{aligned}$$

for $r = 3$. Hence, the definition of $st_1 - LIM^r A_n$ does not coincide with that of $st_2 - LIM^r A_n$.

We are ready to give following inclusion.

Proposition 3.1. If $A \in st_1 - LIM^r A_n$, then $A \subseteq st_2 - LIM^r A_n$.

Proof. Let $A \in st_1 - LIM^r A_n$. On the contrary, suppose that

$$A \not\subseteq st_2 - LIM^r A_n.$$

Then there exists $y \in A$ such that $y \notin st_2 - LIM^r A_n$. Therefore, we have

$$\begin{aligned} y \notin st_2 - LIM^r A_n \quad &\Rightarrow y \notin \bigcap^{c \in \Gamma_{\{A_n\}}} \overline{B}(c,r) \\ &\Rightarrow \exists c \in \Gamma_{\{A_n\}} \therefore y \notin \overline{B}(c,r) \\ &\Rightarrow \rho(y,c) > r \\ &\Rightarrow \inf_{y \in A} \rho(y,c) > r \\ &\Rightarrow d(c,A) \geq r. \end{aligned}$$

Since $A \in st_1 - LIM^r A_n$, we get

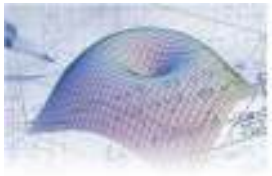
$$\delta\left(\left\{n \in \mathbb{N} \ : \ \left|d(x,A_n) - d(x,A)\right| \geq r + \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$ and each $x \in X$. If we take $x = c$ which is a statistical cluster point, then

$$\delta\left(\left\{n \in \mathbb{N} \ : \ d(c,A_n) < \varepsilon\right\}\right) \neq 0.$$

Because of $d(c,A) \geq r$ and $d(c,A_n) < \varepsilon$ for each $n \in \left\{n \in \mathbb{N} \ : \ d(c,A_n) < \varepsilon\right\}$, we have

$$\delta\left(\left\{n \in \mathbb{N} \ : \ \left|d(c,A_n) - d(c,A)\right| < \varepsilon\right\}\right) = 0. \tag{3.5}$$



This expression (3.5) contradicts to the fact that $A \in st_1 - LIM^r A_n$. This implies that $A \subseteq st_2 - LIM^r A_n$.

Proposition 3.2. If $\{A_{k_n}\}$ is a subsequence of $\{A_n\}$, then

$$st_1 - LIM^r A_n \subseteq st_1 - LIM^r A_{k_n}.$$

Proof. Let $A \in st_1 - LIM^r A_n$. Then, for every $\varepsilon > 0$ and each $x \in X$ there exists an $N(\varepsilon, x) \in \mathsf{N}$ such that

$$\delta\left(\left\{n \in \mathsf{N} : \left|d(x, A_n) - d(x, A)\right| \geq r + \varepsilon\right\}\right) = 0.$$

Since $\{A_{k_n}\}$ is a subsequence of $\{A_n\}$, we have

$$\delta\left(\left\{n \in \mathsf{N} : \left|d(x, A_{k_n}) - d(x, A)\right| \geq r + \varepsilon\right\}\right) = 0.$$

Therefore we get $A \in st_1 - LIM^r A_{k_n}$.

Proposition 3.3. If $r_1 \leq r_2$, then we have

$$st_1 - LIM^{r_1} A_n \subseteq st_1 - LIM^{r_2} A_n \text{ and } st_2 - LIM^{r_1} A_n \subseteq st_2 - LIM^{r_2} A_n.$$

Proof. First we show that $st_1 - LIM^{r_1} A_n \subseteq st_1 - LIM^{r_2} A_n$. Let $r_1 \leq r_2$ and $A \in st_1 - LIM^{r_1} A_n$. Thus, for every $\varepsilon > 0$ and each $x \in X$ we have

$$\delta\left(\left\{n \in \mathsf{N} : \left|d(x, A_n) - d(x, A)\right| \geq r_1 + \varepsilon > r_2 + \varepsilon\right\}\right) = 0.$$

Hence we get $A \in st_1 - LIM^{r_2} A_n$. If $r_1 \leq r_2$ and $c \in \Gamma_{\{A_n\}}$ then we have

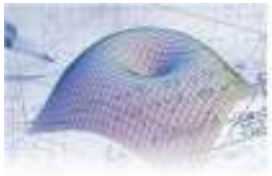
$$\overline{B}(c, r_1) \subseteq \overline{B}(c, r_2)\}$$

$$\bigcap_{c \in \Gamma_{\{A_n\}}} \overline{B}(c, r_1) \subseteq \bigcap_{c \in \Gamma_{\{A_n\}}} \overline{B}(c, r_2).$$

This shows that $st_2 - LIM^{r_1} A_n \subseteq st_2 - LIM^{r_2} A_n$.

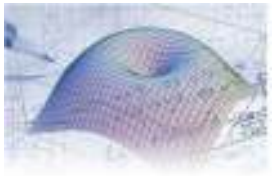
References:

[1] Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Springer, Boston (1990).
[2] Aytar, S.: Rough statistical convergence, Numer. Funct. Anal. and Optimiz. 29(3-4), 291-303 (2008).
[3] Debnath, S., Rakshit, D.: Rough convergence in metric spaces, New Trends in Analysis and Interdisciplinary Applications 449-454 (2017).



[4] Fast, H.: Sur la convergence statistique, Colloq. Math. 2, 241-244 (1951).
[5] Fridy, J.A.: On statistical convergence, Analysis 5, 301-313 (1985).
[6] Fridy, J.A., Orhan, C.: Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125(12), 3625-3631 (1997).
[7] Nuray, F., Rhoades, B.E.: Statistical convergence of sequences of sets, Fasciculi Mathematici 49, 87-99 (2012).
[8] Nuray, F., Dündar, E., Ulusu, U.: Wijsman Statistical Convergence of Double Sequences of Sets, Iranian Journal of Mathematical Sciences and Informatics 16(1), 55-64 (2021).
[9] Ölmez, Ö., Aytar, S.: The relation between rough Wijsman convergence and asymptotic cones, Turkish Journal of Mathematics 40, 1349-1355 (2016).
[10] Ölmez, Ö., Gecit Akçay, F., Aytar, S.: On the rough convergence of a sequence of sets, Electronic Journal of Mathematical Analysis and Applications 10(1), 167-174 (2022).
[11] Phu, H.X.: Rough convergence in normed linear spaces, Numer. Funct. Anal. and Optimiz. 22, 201-224 (2001).
[12] Schoenberg, I.J.: The integrability of certain functions and related summability methods, Amer. Math. Monthly 66, 361-375 (1959).
[13] Subramanian, N., Esi, A.: Wijsman rough convergence of triple sequences, Mat. Stud. 48, 171-179 (2017).
[14] Subramanian, N., Esi, A.: Wijsman rough lacunary statistical convergence on I Cesàro triple sequences, Int. J. Anal. Appl. 16(5), 643-653 (2018).
[15] Wijsman, R.A.: Convergence of sequences of convex sets, cones and functions, Bulletin of the American Mathematical Society 70(1), 186-188 (1964).
[16] Wijsman, R.A.: Convergence of sequences of convex sets, cones and functions II, Trans. Amer. Math. Soc. 123, 32-45 (1966).

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
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Analysis of several ways to formulate the Cayley-Dickson construction

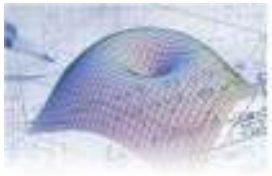
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Abstract The algebraic structure of algebra is obtained by introducing a multiplication operation on a vector space. The elements of vector space expressed in the canonical basis define the algebraic form of the elements of algebra. The Cayley-Dickson construction is a method of obtaining new algebras by doubling the size of the vector space. A sequence of algebras is obtained by the recurrent definition of an application called involution and the definition of multiplication of vectors in doubled algebra relative to the originally considered set. Multiplication is defined as a multiplication of two-dimensional vectors. A matrix writing of multiplication that highlights the role of each element is proposed. Several forms of matrix writing are analyzed. The propagation of the commutativity and associativity properties from the initially considered algebra to the doubled size algebra is analyzed.

Keywords: Cayley-Dickson construction, multiplication matrix writing, switches of factors, imaginary base elements

I. Introduction

The Cayley-Dickson construct is a method of constructing an algebra whose components are two-dimensional vectors with components of a given algebra. Its general formulation for any algebra is presented in [Flaut, 2019]. The sequence of algebras doubled in size was analyzed in the paper [Bulacu, 2009]. The construction of quaternions, octonions, and sedenions in these algebras was shown in [Eakin, 1990], [Bresar, 2011] showed that in the Cayley-Dickson construction a series of unitary algebras is formed in which the subalgebra generated by any scalar element is isomorphic with C.These have been called "locally complex algebras".[Castro, 2020] presented the construction of hypercomplex numbers. [Felipe, 2010] used the Cayley-Dickson construction to construct "relaxed algebras" that do not respect all the properties of algebra, especially the lack of associativity. A generalization of



the construction is proposed in which two parameters are introduced, one in each component and a generalization method in which an algebra is obtained as a direct sum of two distinct algebras.

[Pumplun, 2014] studied how to obtain division algebras by generalizing the duplication process in the Cayley-Dickson construction. This paper analyzes the formulation of the Cayley-Dickson construction.

Although its application is widely used, the meaning of the terms that appear in the multiplications in the definition of multiplication is not highlighted. It is also necessary to correlate the formulation of the Cayley-Dickson construction with the introduction of quaternions, octonions and other hypercomplex numbers in general.

II. Cayley-Dickson construction

Let be A algebra over a field F and a nonzero element from F and be $x \to x^*$ a linear function called involution such that we have:

$$x + x^* = tr(x) \in F \text{ function trace ,} \tag{1}$$

$$xx^* = n(x) \in F \text{ , function norm,} \tag{2}$$

$$(xy)^* = y^* x^* \tag{3}$$

We will consider the direct sum of linear spaces $A^{(2)} = A \oplus A$. On this we have the structure of algebra, A being subalgebra of $A^{(2)}$ through $x \to (x,0)$ and the involution extends to $A^{(2)}$ through:

$$(a_1, a_2)^* = (a_1^*, -a_2) \tag{4}$$

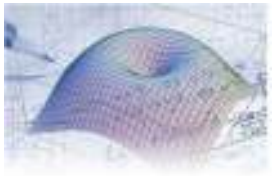
Thus we have:

$$tr(a_1, a_2) = (tr(a_1), 0) \text{ or } tr(a_1, a_2) = tr(a_1), \tag{5}$$

$$n(a_1, a_2) = n(a_1) + \delta n(a_2) \tag{6}$$

,where $\delta \in F$,

We have the following forms of definition of multiplication:



- multiplication on the right, the conjugates of the components of the right element appear:

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - \delta b_2 * a_2, \quad a_2 b_1 * + b_2 a_1)$$

(7)

- multiplication on the right, the conjugates of the components of the left element appear:

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - \delta b_2 a_2 *, \quad a_1 * b_2 + b_1 a_2)$$

(8)

These writings do not highlight the role of each of the four terms. Several similar writings can be considered by switching the terms to both addition and multiplication. Such writing may be considered to represent multiplication in the Cayley-Dickson construction if it respects:

$$(a_1, a_2)(a_1, a_2)^* = (n(a_1, a_2), 0)$$

(9)

To make clear the considerations behind the previous formulas we will consider a matrix writing. One of the two two-dimensional vectors is expanded into a second-order square matrix. Multiplication is the matrix product between a two-dimensional line vector and a second-order square matrix. Thus the result is a two-dimensional line vector. We will consider the following definitions of multiplication:

- Multiplication on the right:

$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta b_2 * & b_1 * \end{pmatrix}$$

(10)

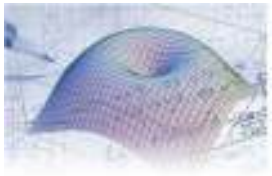
- Multiplication on the left:

$$(a_1, a_2) \bullet (b_1, b_2) = (b_1, b_2) \cdot \begin{pmatrix} a_1 & a_2 \\ -\delta a_2 * & a_1 * \end{pmatrix}$$

(11)

These coincide with the previous ones only if we consider the multiplication of the elements of the commutative matrices. The multiplication of matrices with elements in a non-commutative algebra requires that the order in the multiplication be between the element of the vector with components on the line and the element of the vector with components on the column.

Thus in matrix writing:



$$(a_1, a_2)\cdot\begin{pmatrix} b_1 \\ -\delta\, b_2\, * \end{pmatrix} = a_1\, b_1 - \delta a_2 b_2\, * \neq a_1\, b_1 - \delta b_2\, * a_2$$

(12)

The order of the terms in the sum does not pose problems because the addition is always commutative but must be respected to highlight the meaning of the terms.

Once we chose a form for multiplying two-dimensional vectors, we completed the Cayley-Dickson construction. The Cayley-Dickson construction has the following elements:

- consideration of algebra A ;
- definition of norm, application $n : A \to F$ and function of involution (conjugate number);
- the constant $\delta \in F$ that appears in the definition of multiplication;
- the way of defining the multiplication

This gives a series of algebras doubled in size. A sequence of algebras is formed:

$$A \subset A^{(2)} \subset A^{(4)} \subset A^{(8)} ...$$

(13)

III. Application of the matrix writing of multiplication for a non-commutative algebra

Let be multiplication for an element of algebra $A^{(2)}$ by its conjugate by applying multiplication to the right:

$$(a_1, a_2)\bullet(a, a_2)^* =$$

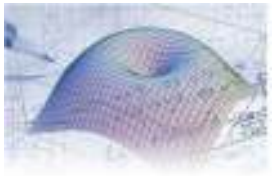
$$(a_1, a_2)\bullet(a_1^*, -a_2) = (a_1, a_2)\cdot\begin{pmatrix} a_1 & -a_2 \\ -\delta(-a_2)^* & (a_1^*)^* \end{pmatrix}$$

$$= (a_1 \;\; a_2)\cdot\begin{pmatrix} a_1^* & -a_2 \\ \delta a_2^* & a_1 \end{pmatrix}$$

$$= (a_1 a_1^* + \delta\, a_2 a_2^* \quad -a_1\, a_2 + a_2 a_1)$$

$$= (n(a_1) + \delta\, n(a_2) \quad -a_1\, a_2 + a_2 a_1) = (n(a_1, a_2), \quad -a_1\, a_2 + a_2 a_1)$$

(14)



We used to $\forall a \in A$, $a \cdot a^* = a^* \cdot a = n(a)$ (2) and $(a^*)^* = a$

(15)

We notice that this respects the condition (9) if and only if the multiplication in is commutative. This condition is not met for non-commutative algebra so we must intervene in the definition of the multiplication. This was done for the relations (7) but it do not respect the order of the factors in the matrix multiplication. In order to have both the correctness of the relations and the advantage of matrix writing, we introduce a switching operator in the matrix writing. It acts on the element it "multiplies on right" and switches the order of factors into multiplication. We will write:

$$a \cdot (\tilde{r} b) \overset{def}{=} b \cdot a \quad \forall a,b \in A$$

(16)

Thus the relation:

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - \delta b_2^* a_2, \quad a_2 b_1^* + b_2 a_1)$$

(17)

is written in matrix multiplication:

$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & \tilde{r} b_2 \\ -\delta \tilde{r} b_2^* & b_1^* \end{pmatrix}$$

(18)

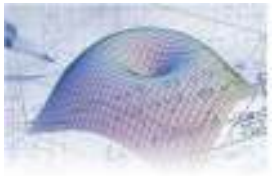
The previous demonstration shows that the switching operator on the first column has no effect on determining the norm formula. Its presence comes from a logical association between the switching operator and the conjugation operator. Thus we can give up the switching operator if we consider that the multiplication of the conjugate element when it is the second element of the multiplication also achieves the switching of the multiplication. We have:

$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta \tilde{r} b_2^* & \tilde{r} b_1^* \end{pmatrix}$$

$$\overset{def}{=} (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta\, b_2(\tilde{r}^*) & b_1(\tilde{r}^*) \end{pmatrix} \qquad (19)$$

This means a composition of the two operators.

In the following we will eliminate unnecessary switching on the first column and we will establish the formula for multiplication as:



$$(a_1,a_2)\bullet(b_1,b_2)\overset{def}{=}(a_1,a_2)\cdot\begin{pmatrix} b_1 & b_2 \\ -\delta b_2^* & \tilde{r}\, b_1^* \end{pmatrix}=(a_1b_1-\delta a_2b_2^*,\quad a_1b_2+b_1^*a_2)$$

(20)

IV. Checking the properties of multiplication

Using the form of multiplication defined in the previous relation we will check the properties of multiplication.

Let's check the distributivity:

- distributivity to the right:

$$(a_1,a_2)\bullet[(b_1,b_2)+(b_3,b_4)]=(a_1,a_2)\cdot\begin{pmatrix} b_1+b_3 & b_2+b_4 \\ -\delta(b_2+b_4)^* & \tilde{r}\,(b_1+b_3)^* \end{pmatrix}=$$

$$(a_1,a_2)\cdot\begin{pmatrix} b_1 & b_2 \\ -\delta b_2^* & \tilde{r}\, b_1^* \end{pmatrix}+(a_1,a_2)\cdot\begin{pmatrix} b_3 & b_4 \\ -\delta b_4^* & \tilde{r}\, b_3^* \end{pmatrix}=$$

$(a_1,a_2)\bullet(b_1,b_2)+\ (a_1,a_2)\bullet(b_3,b_4)$

(21)

We used that: $(x+y)^*=x^*+y^*\quad \forall x,y\in A$

(22)

$\tilde{r}(x+y)=\tilde{r}x+\tilde{r}y$

(23)

- distributivity to the left

$$[(a_1,a_2)+(a_3,\,a_4)]\bullet(b_1,b_2)=(a_1+a_3,a_2+a_4)\cdot\begin{pmatrix} b_1 & b_2 \\ -\delta b_2^* & \tilde{r}\, b_1^* \end{pmatrix}=$$

$(a_1,a_2)\bullet(b_1,b_2)+(a_3,a_4)\bullet(b_1,b_2)$

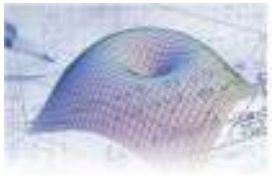
(24)

The distribution of the multiplication of the matrices relative to the sum of the matrices is used.

Let's check the multiplication of an element with its conjugate:

$(a_1,a_2)\bullet(a_1,a_2)^*=$

$$(a_1,a_2)\bullet(a_1^*,-a_2)=(a_1,a_2)\cdot\begin{pmatrix} a_1 & -a_2 \\ -\delta(-a_2)^* & \tilde{r}(a_1^*)^* \end{pmatrix}$$



$$= (a_1 \;\; a_2) \cdot \begin{pmatrix} a_1^* & -a_2 \\ \delta a_2^* & \tilde{r}\, a_1 \end{pmatrix} = (a_1 a_1^* + \delta\, a_2 a_2^*, \quad -a_1\, a_2 + a_1 a_2) = (n(a_1, a_2),\, 0)$$

(25)

$$(a_1, a_2)^* \bullet (a_1, a_2) = (a_1^*, -a_2) \bullet (a_1, a_2) = (a_1^* - a_2) \cdot \begin{pmatrix} a_1 & a_2 \\ -\delta\, a_2^* & \tilde{r}\, a_1^* \end{pmatrix}$$

$$= (a_1^* a_1 + \delta\, a_2 a_2^*,\; a_1^* a_2 - a_1^* a_2) = (n(a_1) + \delta\, n(a_2),\; a_1^* a_2 - a_1^* a_2)$$

$$= (n(a_1, a_2),\, 0)$$

(26)

This shows that the multiplication thus defined respects the condition (6)

Let's calculate the square of an element of algebra $A^{(2)}$.

$$(a_1, a_2) \bullet (a_1, a_2) = (a_1, a_2) \cdot \begin{pmatrix} a_1 & a_2 \\ -\delta\, a_2^* & \tilde{r}\, a_1^* \end{pmatrix} = (a_1 a_1 - \delta a_2 a_2^*, \quad a_1 a_2 + a_1^*$$

$$= (a_1^{\,2} - \delta n(a_2), \quad tr(a_1) a_2)$$

(27)

This relation can be applied particularly for the elements that we will call in the following imaginary elements of the base, respectively real elements of the base.

V. Particular applications

Proposition 1 If algebra is commutative and involution is the identical function then algebra is commutative.

Let be commutative algebra A. We will study the commutativity of algebra $A^{(2)}$.

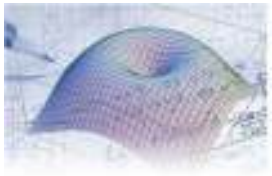
$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta\, b_2^* & \tilde{r}\, b_1^* \end{pmatrix} = (a_1 b_1 - \delta a_2 b_2^*, \quad a_1 b_2 + b_1^* a_2)$$

(28)

$$(b_1, b_2) \bullet (a_1, a_2) = (b_1, b_2) \cdot \begin{pmatrix} a_1 & a_2 \\ -\delta\, a_2^* & \tilde{r}\, a_1^* \end{pmatrix} = (b_1 a_1 - \delta b_2 a_2^*, \quad b_1 a_2 + a_1^* b_2)$$

(29)

Equating the two relations and applying the commutativity in A results:



$$a_2 b_2 * = b_2 a_2 *$$

(30)

$$a_1 b_2 + b_1 * a_2 = b_1 a_2 + a_1 * b_2$$

(31)

The above relations are true for any elements in A if the involution is the identical function and multiplication in A is commutative. This is the case of the algebra of complex numbers obtained as a doubled algebra of the algebra of real numbers.

Another idea that emerges in the writing of the Cayley-Dickson construct is to use the trace and norm functions instead of the conjugation operator. The definition of multiplication according to the trace application is as follows:

$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta(tr(b_2) - b_2) & \tilde{r}(tr(b_1) - b_1) \end{pmatrix} =$$

$$(a_1, a_2) \cdot \begin{pmatrix} 0 & 0 \\ -\delta\, tr(b_2) & \tilde{r}\, tr(b_1) \end{pmatrix} + (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ \delta\, b_2 & \tilde{r}\, b_1 \end{pmatrix}$$

(32)

The second term represents the multiplication of algebraic forms associated with two-dimensional vectors. It is thus shown that if $tr(b_1) = tr(b_2) = 0$ then the multiplication formulated in the Cayley-Dickson construction is an algebraic multiplication. This happens for elements considered purely imaginary. The definition of multiplication according to the norm application is as follows:

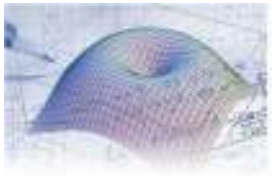
$$(a_1, a_2) \bullet (b_1, b_2) = (a_1, a_2) \cdot \begin{pmatrix} b_1 & b_2 \\ -\delta n(b_2) b_2^{-1} & \tilde{r}\, n(b_1) b_1^{-1} \end{pmatrix}$$

(33)

Particular applications of this formula are obtained for $n(b_1) = n(b_2) = 1$.

VI. Evaluation of the multiplication of the imaginary elements of the base from $A^{(2)}$

Definition 2 A subset I of algebra A, $I \subset A$, called the imaginary elements of the basis of algebra A has the following property:



$$p^* = -p, \forall\, p \in I$$

(34)

Consequence 3 We will have: $tr(p) = p + p^* = 0$ and $n(p) = p \cdot p^* = -p^2 \in F$.

Proposition 4 $\forall\, p \in I$, elements $(p, 0)$ and $(0, p) \in I^{(2)} \subset A^{(2)}$, are imaginary elements of the base in $A^{(2)}$.

Proof:

$$(p,0)^* = (p^*,0) = (-p, 0) = -(p,0)$$

(35)

$$(0, p)^* = (0,-p) = -(0, p)$$

(36)

Thus, we observe that the definition of the imaginary elements of the base in $A^{(2)}$ is respected.

Let be the multiplication of the imaginary elements of the base. We have the following cases:

1) $$(p,0) \bullet (q,0) = (p,0) \cdot \begin{pmatrix} q & 0 \\ -\delta \cdot 0 & \tilde{r}\,(-q) \end{pmatrix} = (pq,\ 0)$$

(37)

2) $$(p,0) \bullet (0,q) = (p,0) \cdot \begin{pmatrix} 0 & q \\ -\delta \cdot (-q) & \tilde{r} \cdot 0 \end{pmatrix} = (0,\ pq)$$

(38)

3) $$(0,p) \bullet (q,0) = (0,p) \cdot \begin{pmatrix} q & 0 \\ -\delta \cdot 0 & \tilde{r}\,(-q) \end{pmatrix} = (0,\ -qp)$$

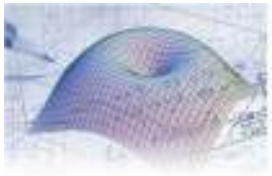
(39)

4) $$(0,p) \bullet (0,q) = (0,p) \cdot \begin{pmatrix} 0 & q \\ -\delta \cdot (-q) & \tilde{r} \cdot 0 \end{pmatrix} = (\delta pq,\ 0)$$

(40)

These relations can be put in the form of a table:

Table 1 Multiplication of vector imaginary base elements from $A^{(2)}$ build with one component as imaginary element in A.



	(q,0)
(0,q)	
	(pq,0)
p,0)	(0, pq)
	(0,-qp)
0,p)	(δ pq,0)

For $p = q$ we will have:

Table 2 Application of Table 1 for $p = q$

	(p,0)
(0,p)	
	(p^2,0)
p,0)	(0, p^2)
	(0,-p^2)
0,p)	(δ p^2,0)

Even if at first look it seems that table 2 has elements from F^2 because $p^2 \in F$, this is not true. In the vector components the scalar element appears multiplied by the unit element from A.

To multiply the imaginary elements of the base of algebra A, two hypotheses are formulated, either this multiplication is commutative or this multiplication is anticommutative.

Proposition 5 Multiplications of type $(p,0)\bullet(q,0)$ and $(0,p)\bullet(0,q)$ keep the commutative or anticommutative property of algebra A. But multiplications type $(p,0)\bullet(0,q)$ are anticomutative no matter the multiplication of A.

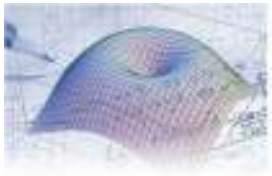
Proof: $(p,0)\bullet(0,q) = (0,\ pq)$; $(0,q)\bullet(p,0) = (0,-pq) = -(0,\ pq)$.

Let's analyze the associativity of the elements of the set $I^{(2)}$. We will calculate:

1) $(a,0)\bullet[(b,0)\bullet(c,0)] = (a,0)\bullet(bc,0) = (abc,0)$ (41)

$$[(a,0)\bullet(b,0)]\bullet(c,0)] = (ab,0)\bullet(c,0) = (abc,0) \quad (42)$$

2) $(0,a)\bullet[(b,0)\bullet(c,0)] = (0,a)\bullet(bc,0) = (0,-bca)$ (43)



$$[(0,a)\bullet(b,0)]\bullet(c,0)] = (0,-ba)\bullet(c,0) = (0,\ -c(-ba)) = (0,cba) \tag{44}$$

3) $$(a,0)\bullet[(0,b)\bullet(c,0)] = (a,0)\bullet(0,-cb) = (0,-acb) \tag{45}$$

$$[(a,0)\bullet(0,b)]\bullet(c,0) = (0,ab)\bullet(c,0) = (0,-cab) \tag{46}$$

4) $$(a,0)\bullet[(b,0)\bullet(0,c)] = (a,0)\bullet(0,\ bc) = (0,abc) \tag{47}$$

$$[(a,0)\bullet(b,0)]\bullet(0,c) = (ab,0)\bullet(0,\ c) = (0,abc) \tag{48}$$

5) $$(0,a)\bullet[(0,b)\bullet(c,0)] = (0,a)\bullet(0,-cb) = (-\delta acb,0) \tag{49}$$

$$[(0,a)\bullet(0,b)]\bullet(c,0) = (\delta ab,0)\bullet(c,0) = (\delta abc,0) \tag{50}$$

6) $$(0,a)\bullet[(b,0)\bullet(0,c)] = (0,a)\bullet(0,bc) = (\delta abc,0) \tag{51}$$

$$[(0,a)\bullet(b,0)]\bullet(0,c) = (0,-ba)\bullet(0,c) = (-\delta bac,0) \tag{52}$$

7) $$(a,0)\bullet[(0,b)\bullet(0,c)] = (a,0)\bullet(\delta bc,0) = (\delta abc,0) \tag{53}$$

$$[(a,0)\bullet(0,b)]\bullet(0,c) = (0,ab)\bullet(0,c) = (\delta abc,0) \tag{54}$$

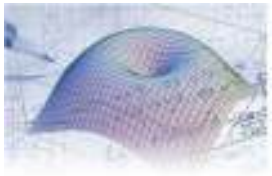
8) $$(0,a)\bullet[(0,b)\bullet(0,c)] = (0,a)\bullet(\delta bc,0) = (0,-\delta bca) \tag{55}$$

$$[(0,a)\bullet(0,b)]\bullet(0,c) = (\delta ab,0)\bullet(0,c) = (0,-\delta bca) \tag{56}$$

These relations show the following:

- Regardless of the association mode, the form of the resulting vector is kept;
- There are particular cases in which multiplication is associative;
- Where equality is not achieved, this could be achieved by reversing two factors. Based on the previous relations we formulate the following property.

Property 6 If multiplying the imaginary base elements in algebra A is anticommutative, then multiplication of imaginary base elements in algebra $A^{(2)}$ is associative.



VII. Evaluation of the multiplication of the real elements of the base from $A^{(2)}$

Definition 7 It is considered a subset $U \subset A$ of algebra A elements, called the real base units with property:

$$u^* = u \qquad \forall u \in U \tag{57}$$

Proposition 8 If $u \in U \subset A$ then $(u,\ 0) \in U^{(2)} \subset A^{(2)}$.

Proof: $(u,0)^* = (u^*,0) = (u,0)$

Let be the multiplication of two real units of the base:

$$(u,0) \bullet (v,0) = (u,0) \cdot \begin{pmatrix} v & 0 \\ 0 & \widetilde{r}\, v \end{pmatrix} = (uv,\ 0) = (v,0) \bullet (u,0) \tag{58}$$

It is shown that the multiplication of the real elements of the base is always commutative.

Observation 9 $(0,u) \in I^2 \subset A^{(2)}$ ie is the imaginary unit

Proof : We have: $(0,u)^* = (0,-u)$, and hence $(0,u)^* + (0,u) = (0,0)$

Let's analyze multiplication between real unity and imaginary units of the base:

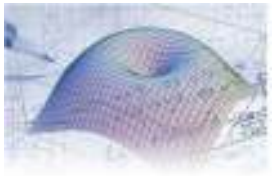
1)
$$(u,0) \bullet (q,0) = (u,0) \cdot \begin{pmatrix} q & 0 \\ -\delta \cdot 0 & \widetilde{r}\,(-q) \end{pmatrix} = (uq,\ 0) \tag{59}$$

$$(p,0) \bullet (u,0) = (p,0) \cdot \begin{pmatrix} u & 0 \\ -\delta \cdot 0 & \widetilde{r}\, u \end{pmatrix} = (pu,\ 0) \tag{60}$$

2)
$$(u,0) \bullet (0,q) = (u,0) \cdot \begin{pmatrix} 0 & q \\ -\delta \cdot (-q) & \widetilde{r} \cdot 0 \end{pmatrix} = (0,\ uq) \tag{61}$$

$$(0,p) \bullet (u,0) = (0,p) \cdot \begin{pmatrix} u & 0 \\ 0 & \widetilde{r} \cdot u \end{pmatrix} = (0,\ up) \tag{62}$$

It is noted that the multiplication of real base units with imaginary units is commutative.



VIII. Conclusions

A particular form of writing Cayley-Dickson was proposed.This has written the multiplication of two-dimensional vectors by expanding one of the vector in a square matrix. Thus, the multiplication of two-dimensional vectors was written as a matrix product.To show the order of factors in the matrix multiplication a switch operator was introduced. Thus, this write is also applicable for non-commutative algebras.

A form of writing multiplication in the Cayley-Dickson construction that has a single commutation between factors has been chosen. Matrix writing of multiplication

provides an important tool in assessing the transmission of properties from algebra A elements to algebra $A^{(2)}$ elements.

The distribution of multiplication for the addition was checked. The multiplication formula was written using the functions trace and norm, instead of using the involution function (the conjugate).

Among algebra A elements have been identified imaginary base elements and real elements of the base according to the particular definitions of the involution function (the conjugate).

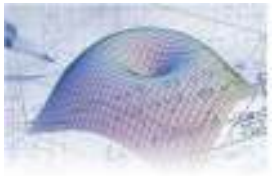
This classification was made without identifying a basis in A. The two-dimensional vectors were built in $A^{(2)}$ for which one components are these elements and the other is null. It has been shown that relative to the definitions of the conjugate considered in $A^{(2)}$ two-dimensional vectors built maintained their classification as imaginary base elements or real base elements. Using the multiplication formula, the multiplication of these vectors was studied. The commutativity and associativity of these multiplications has been studied.

References

1.[Flaut, 2019] Cristina Flaut, Diana Savin, Some remarks regarding *l*- elements defined in algebras obtained by the Cayley–Dickson process Chaos, Solitons and Fractals 118 (2019) p:112–116.

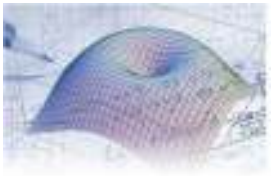
2.[Bulacu, 2009] Daniel Bulacu, The weak braided Hopf algebra structure of some Cayley–Dickson algebras Journal of Algebra 322 (2009) p:2404–2427.

3.[Eakin, 1990] Paul Eakin and Avinash Sathaye, On Automorphisms and Derivations of
Cayley-Dickson Algebras Journal of Algebra 129, p:263-278 (1990).



4.[Bresar, 2011] Matej Brešar Peter Šemrl Špela Špenko, On locally complex algebras and low-dimensional Cayley–Dickson algebras Journal of Algebra 327 (2011) , p:107–125.
5.[Castro, 2020] Fidelis Zanetti de Castro, Marcos Eduardo Valle, A broad class of discrete-time hypercomplex-valued Hopfield neural networks Neural Networks 122 (2020) p:54–67.
6.[Felipe, 2010] Ra´ul Felipe, Ra´ul Felipe-Sosa Relaxed linear spaces and a generalization of the
Cayley–Dickson processLinear Algebra and its Applications 434, (2011), p:424–442.
7.[Pumplun, 2014] S.Pumplün How to obtain division algebras from a generalized Cayley–Dickson doubling process Journal of Algebra 402 (2014) p:406–434.

First Western Balkan Conference on Mathematics and Applications 10-12-June 2021, Prishtine, Kosovo
(http://fwbcma2021.ilirias.com/)



ROUGH LACUNARY STATISTICAL CONVERGENCE OF SEQUENCES OF FUZZY NUMBERS

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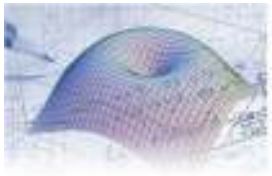
In this paper, the concept of rough lacunary statistical convergence of sequences of fuzzy numbers is introduced. Also, the set of rough lacunary statistical limit points of such sequences is defined. Then, some properties of this set are proved by using similar proof techniques in [6]. Finally, the relations between the set of lacunary statistical cluster points and the set of rough lacunary statistical limit points of a sequence of fuzzy numbers are investigated.

Keywords: Lacunary statistical convergence, Lacunary statistical cluster point, Lacunary statistical limit point, Rough statistical convergence, Sequence of fuzzy numbers.

Introduction

The concept of statistical convergence was defined for a complex number sequence by Fast [8] as follows:

A sequence of x is said to be statistically convergent to L, if for each $\varepsilon > 0$,



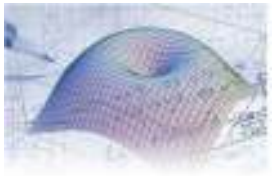
$$\lim_n \frac{1}{n}\left|\left\{ k \le n : \left|x_k - L\right| \ge \varepsilon \right\}\right| = 0 .$$

In this definition, by replacing the set $\{k : k \le n\}$ with $\{k : k_{s-1} < k \le k_s\}$ for some lacunary sequence $\{k_s\}$, Fridy and Orhan [9] defined lacunary statistical convergence. Also, they linked this concept with the summability theory. Then, the idea was extended to sequences of fuzzy numbers by Nuray [13], then studied by Kwon et. al. [12]. Many authors have worked on this concept and its generalizations. For examples, Altın, Et and Çolak [2] defined lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers, Savaş [15] defined lacunary statistically convergent double sequences of fuzzy numbers, Gökhan, Et and Mursaleen [10] studied almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers.

In 2001, Phu [14] brought a different perspective to convergence called rough convergence by expressing that the limit may not be unique for roughness degree in finite dimensional normed spaces and thus the rough limit set can be defined. Then, Aytar [3] introduced the statistical analogue of this new convergence. He defined the set of rough statistical limit points of a sequence and gave two statistical convergence criteria associated with the set. Also,

Akçay and Aytar [1] studied rough convergence of sequences of fuzzy numbers. Babaarslan and Tuncer [4] generalized rough convergence for double sequences of fuzzy numbers. Moreover, Debnath and Rakshit [6] introduced rough statistical convergence of sequences of fuzzy numbers.

Recently, the ideas of rough statistical convergence and lacunary convergence has been combined by many authors like as follows:



Kişi and Dündar [11] gave rough I_2-lacunary statistical convergence of double sequences, Das et. al. [5] studied rough lacunary statistical convergence of a sequence of random variables in probability. Debnath and Subramanian [7] studied on generalized rough lacunary statistical triple difference sequence spaces in probability of fractional order defined by Musielak-Orlicz function.

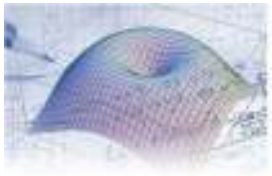
In this work, we define the concept of rough lacunary statistical convergence in the space of fuzzy number sequences that are not a normed space. In addition, the rough lacunary statistical limit set is expressed and the properties of this set are mentioned. Moreover, the concepts of lacunary statistical limit inferior and limit superior are defined and their relationship with the rough statistical limit set is shown. Also, the relationship between lacunary statistical convergence and rough lacunary statistical convergence is given.

Preliminaries

A fuzzy number is a function X from $\mathbb{R}$ to $[0,1]$, satisfying
X is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $X(x_0)=1$;
X is fuzzy convex, i.e. for any $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$
$X(\lambda x+(1-\lambda)y) \ge \min\{X(x), X(y)\}$
X is upper semi-continuous;
the closure of $X^0 = \{x \in \mathbb{R} : X(x) > 0\}$ is compact.

These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$level set

$X^\alpha := \{x \in \mathbb{R} : X(x) \ge \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$ is a nonempty compact convex subsets of $\mathbb{R}$. The set of all fuzzy numbers is denoted by $L(\mathbb{R})$. The supremum metric on $L(\mathbb{R})$ is defined by $\overline{d}(X,Y) = \sup_{0\le\alpha\le 1} \max\left\{\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right\}$ and $\left(L(\mathbb{R}), \overline{d}\right)$ is a



metric space. A lacunary sequence $\theta = (k_s)$ is an increasing sequence of integers such that $k_0 = 0$, $h_s = k_s - k_{s-1} \to \infty$, as $s \to \infty$ and the interval determined by this sequence is denoted by $I_s = (k_{s-1}, k_s]$.

The lacunary density of K is defined as $\delta_\theta(K) = \lim_s \frac{|I_s \cap K|}{h_s}$, [9].

A sequence of $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically convergent to a fuzzy number X_0, if for every $\varepsilon > 0$,

$$\delta_\theta(K(\varepsilon)) = \lim_s \frac{1}{h_s} \left| \left\{ k \in I_s : \bar{d}(X_k, X_0) \geq \varepsilon \right\} \right| = 0,$$

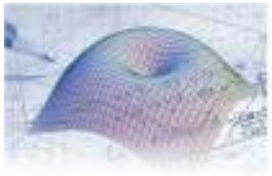
where $K(\varepsilon) = \left\{ k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon \right\}$, [13].

Throughout this paper, we assume that $X = (X_k)$ is a sequence of fuzzy numbers, $\theta = (k_s)$ is a lacunary sequence and r is a non-negative real number.

Rough Lacunary Statistical Convergence of Sequences of Fuzzy Numbers

Definition 3.1. A sequence of $X = (X_k)$ is said to be rough lacunary statistical convergent to a fuzzy number X_*, written as $X_k \overset{rs_\theta}{\to} X_*$, if for every $\varepsilon > 0$,

$$\lim_s \frac{1}{h_s} \left| \left\{ k \in I_s : \bar{d}(X_k, X_*) \geq r + \varepsilon \right\} \right| = 0. \qquad (1)$$



Here, r is called the roughness degree. If we take $r = 0,$ we obtain the lacunary statistical convergence.

In a similar fashion to the idea of rough convergence of sequence of fuzzy numbers, we can interpreted our idea as follows:

Assume that $Y = (Y_k)$ is lacunary statistically convergent to a fuzzy number X_*. Let $Y = (Y_k)$ cannot be determined exactly for every $k \in I_s$. If it is difficult to calculate with the elements of $Y = (Y_k)$, then we obtain a triangular fuzzy number sequences (named approximated sequence) $X = (X_k)$ satisfying $\bar{d}(X_k, Y_k) \le r$. So, we can use approximate value of $Y = (Y_k)$ for calculated simply. In here, $X = (X_k)$ is not lacunary statistical convergent but as the inclusion

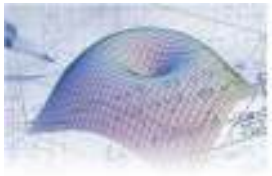
$$\left\{k \in I_s : \bar{d}(Y_k, X_*) \ge \varepsilon\right\} \supseteq \left\{k \in I_s : \bar{d}(X_k, X_*) \ge r + \varepsilon\right\}$$

holds and from our assumption we get the sequence $X = (X_k)$ is rough lacunary statistical convergent to X_*. The rough lacunary statistical limit of a sequence may not be unique for $r > 0$. So, we define the $r_\theta -$statistical limit set as

$$S_\theta - LIM^r X_k := \left\{X_* \in L(R) : X_k \overset{rs_\theta}{\to} X_*\right\}.$$

Now, we introduce $S_\theta -$ limit superior and $S_\theta -$ limit inferior of X. Then, we give the relationship between these concepts and $r_\theta -$statistical limit set of X.

Definition 3.2. If $X = (X_k)$ is a $S_\theta -$bounded sequences of fuzzy numbers, then rough lacunary statistical limit inferior of X is given by



$$S_\theta - \liminf_{k\to\infty} X_k := \inf A_X,$$

and rough lacunary statistical limit superior of X is given by

$$S_\theta - \limsup_{k\to\infty} X_k := \sup B_X,$$

where

$$A_X := \left\{\mu \in L(\mathbb{R}) : \delta_\theta\left(\left\{k \in N : X_k \succ M\right\}\right) \neq 0\right\},$$

$$B_X := \left\{\mu \in L(\mathbb{R}) : \delta_\theta\left(\left\{k \in N : X_k \prec M\right\}\right) \neq 0\right\}.$$

If $S_\theta - LIM^r X_k \neq \varnothing$ for a sequence $\left(X_k\right)$ of fuzzy numbers, then we have

$S_\theta - LIM^r X_k \subseteq \left[S_\theta - \limsup X_k - r, S_\theta - \liminf X_k + r\right]$.

To prove the Theorem 3.1, we first give Lemma 3.1.

Lemma 3.1. If $X_* \in S_\theta - LIM^r X_k$, then we have

$$\delta_\theta\left(\left\{k \in \mathbb{N} : \overline{d}\left(S_\theta - \limsup X_k, X_*\right) > r\right\}\right) = 0$$

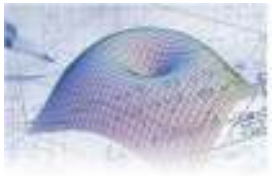
and

$$\delta_\theta\left(\left\{k \in \mathbb{N} : \overline{d}\left(S_\theta - \liminf X_k, X_*\right) > r\right\}\right) = 0.$$

Proof. We assume that $\delta_\theta\left(k \in \mathbb{N} : \overline{d}\left(S_\theta - \limsup X_k, X_*\right) > r\right) \neq 0$. From this assumption, we can define $\tilde{\varepsilon} > 0$ as below

$$\tilde{\varepsilon} := \frac{\overline{d}\left(S_\theta - \limsup X_k, X_*\right) - r}{2}.$$

By definition of $S_\theta -$ limit superior, we have $\overline{d}\left(S_\theta - \limsup X_k, X_k\right) < \tilde{\varepsilon}$ for each $k \in K_1$, where $\delta_\theta\left(K_1\right) \neq 0$.



Also, since $X_* \in S_\theta - LIM^r X_i$, we have $\overline{d}\left(X_k, X_*\right) \geq r + \tilde{\varepsilon}$ for each $k \in K_2$, where $\delta_\theta\left(K_2\right) = 0$.

Let $K := K_1 \bigcap K_2^c$. Then $\delta_\theta\left(K\right) \neq 0$.

Hence,

$$\overline{d}\left(S_\theta - \limsup X_k, X_*\right) \leq \overline{d}\left(S_\theta - \limsup X_k, X_k\right) + \overline{d}\left(X_k, X_*\right) < r + \tilde{\varepsilon}$$

$$= r + \overline{d}\left(S_\theta - \limsup X_k, X_*\right) - r = \overline{d}\left(S_\theta - \limsup X_k, X_*\right)$$

for each $k \in K$. This contradiction proves the Lemma 3.1.

Theorem 3.1. If $S_\theta - LIM^r X_k \neq \varnothing$ for a sequence $\left(X_k\right)$, then we have

$$S_\theta - LIM^r X_k \subseteq \left[S_\theta - \limsup X_k - r, S_\theta - \liminf X_k + r\right].$$

Proof. Let $\varepsilon > 0$. Assume that $X_* \in S_\theta - LIM^r X_k$. Then,

$$\delta_\theta\left(\left\{k \in I_s : \overline{d}\left(X_k, X_*\right) \geq r + \varepsilon\right\}\right) = 0.$$

That is, for each $k \in K_1$, where $\delta_\theta\left(K_1\right) \neq 0,$ we can write

$$\sup_{0 \leq \alpha \leq 1} \max\left\{\left|\underline{X}^\alpha - \underline{X_*}^\alpha\right|, \left|\overline{X}^\alpha - \overline{X_*}^\alpha\right|\right\} < r + \varepsilon$$

or

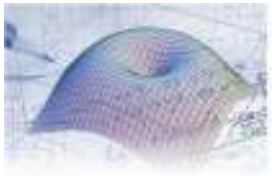
$$\sup_{0 \leq \alpha \leq 1}\left|\underline{X}^\alpha - \underline{X_*}^\alpha\right| < r + \varepsilon \text{ and } \sup_{0 \leq \alpha \leq 1}\left|\overline{X}^\alpha - \overline{X_*}^\alpha\right| < r + \varepsilon.$$

Hence, for each $k \in K_1$, where $\delta_\theta\left(K_1\right) \neq 0$, we have

$$X_* - (r_1 + \varepsilon) \prec X_k \prec X_* + (r_1 + \varepsilon).$$

In here, we obtain two conditions:

First, we get $S_\theta\left(k : X_k \prec X_* + (r_1 + \varepsilon)\right) \neq 0$, i.e, $X_* + (r_1 + \varepsilon) \in B_X$ and



according to the Lemma 3.1, we write

$$S_\theta - \limsup X_k := \sup B_x = \mu \le X_* + (r_1 + \varepsilon).$$

Secondly, we get $S_\theta\left(\{k : X_k \succ X_* - (r_1 + \varepsilon_1)\}\right) \neq 0$, i.e, $X_* - (r_1 + \varepsilon) \in A_X$ and according to the Lemma 3.1, we write

$$S_\theta - \liminf X_k := \inf A_x = \eta \ge X_* - (r_1 + \varepsilon_1).$$

Hence, we obtain $X_* \in \left[S_\theta - \limsup X_k - r, S_\theta - \liminf X_k + r\right]$.

Theorem 3.2. For a sequence $\left(X_k\right)$, we have $diam\left(S_\theta - LIM^r X_k\right) \le 2r$.

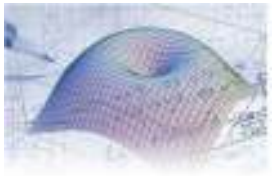
Proof. Assume that $diam\left(S_\theta - LIM^r X_k\right) > 2r.$ Then, there exists $Y,\ Z \in S_\theta - LIM^r X$ provided that $\overline{d}(Y,Z) > 2r$. Let $\varepsilon \in \left(0, \frac{\overline{d}\left(Y,Z\right)}{2} - r\right)$. Since $Y,\ Z \in S_\theta - LIM^r X$, we have $\delta_\theta(K_1) = 0$ and $\delta_\theta(K_2) = 0$ for $\forall \varepsilon > 0$, where $K_1 := \left\{k \in I_s : \overline{d}(X_k, Y) \ge r + \varepsilon\right\}$ and $K_2 := \left\{k \in I_s : \overline{d}(X_k, Z) \ge r + \varepsilon\right\}$.

Thus, for all $k \in K_1^{\,c} \bigcap K_2^{\,c}$, we get

$$\overline{d}\left(Y,Z\right) \le \overline{d}\left(X_k,Y\right) + \overline{d}\left(X_k,Z\right) < 2\left(r+\varepsilon\right) = \overline{d}\left(Y,Z\right).$$

So, it is a contradiction.

Theorem 3. 3. The r_θ – statistical limit set of $X = \left(X_k\right)$ is closed.



Proof. Let (Y_k) be a sequence in $S_\theta - LIM^r X_k$ which converges to Y_*. If we show that $Y_* \in S_\theta - LIM^r X_k$, then the proof will be completed.

Let $\varepsilon > 0$ be given. Since (Y_k) converges to Y_*, there exists $k_{\frac{\varepsilon}{2}} \in I_n$ such that

$\bar{d}(Y_i, Y_*) < \frac{\varepsilon}{2}$ for all $k > k_{\frac{\varepsilon}{2}}$.

Now, we choose a $k_0 \in I_n$ such that $k_0 > k_{\frac{\varepsilon}{2}}$. Then we can write $\bar{d}(Y_{k_0}, Y_*) < \frac{\varepsilon}{2}$.

Since

$(Y_k) \subseteq S_\theta - LIM^r X_k$, we write $(Y_{k_0}) \subseteq S_\theta - LIM^r X_k$,

Namely

$$\left\{k \in I_s : \bar{d}(X_k, Y_*) < r + \varepsilon\right\} \supseteq \left\{k \in I_s : \bar{d}(X_k, Y_{k_0}) < r + \frac{\varepsilon}{2}\right\}$$

$$\lim_s \frac{1}{h_s} \left|\left\{k \in I_s : \bar{d}\left(X_k, Y_{k_0}\right) \geq r + \tfrac{\varepsilon}{2}\right\}\right| = 0. \qquad (2)$$

Let us show that the inclusion holds.

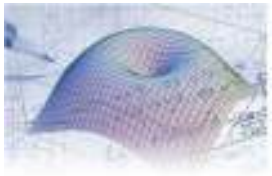
For $i \in \left\{k \in I_s : \bar{d}(X_k, Y_{k_0}) < r + \frac{\varepsilon}{2}\right\}$, we have $\bar{d}(X_i, Y_{k_0}) < r + \frac{\varepsilon}{2}$. Since

$$\bar{d}(X_i, Y_*) < \bar{d}(X_i, Y_{k_0}) + \bar{d}(Y_{k_0}, Y_*) < r + \varepsilon,$$

we have $i \in \left\{k \in I_s : \bar{d}(X_k, Y_*) < r + \varepsilon\right\}$.

From Eq. (2), we can write

$$\lim_s \frac{1}{h_s} \left|\left\{k \in I_s : \bar{d}\left(X_k, Y_{k_0}\right) < r + \frac{\varepsilon}{2}\right\}\right| = 1,$$



so we have

$$\lim_s \frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(X_k, Y_*\right) < r+\varepsilon\right\}\right| = 1,$$

which completes the proof.

Theorem 3.4. A sequence of $\left(cX_k\right)$ is $r-$lacunary statistical convergent to cX_*, for any $c \in \mathbb{R}$, if $X = \left(X_k\right)$ is $\frac{r}{|c|}-$lacunary statistical convergent to X_*.

Proof. Given $\varepsilon > 0$, we write the inclusion

$$\frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(cX_k, cX_*\right) \geq r+\varepsilon\right\}\right| \leq \frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(X_k, X_*\right) \geq \frac{(r+\varepsilon)}{|c|}\right\}\right|,$$

by taking limit $s \to \infty$, the proof is completed.

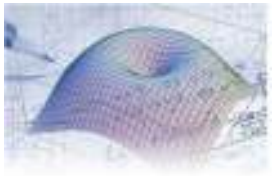
Theorem 3.5. If $X = \left(X_k\right)$ is r_1-lacunary statistically convergent to X_* and $Y = \left(Y_k\right)$ is r_2-lacunary statistically convergent to Y_*, then $(X_k + Y_k)$ is $(r_1 + r_2)-$lacunary statistically convergent to $X_* + Y_*$.

Proof. Given $\varepsilon > 0$, we can write

$$\frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(X_k + Y_k, X_* + Y_*\right) \geq (r_1 + r_2) + \varepsilon\right\}\right| \leq$$

$$\frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(X_k, X_*\right) \geq r_1 + \frac{\varepsilon}{2}\right\}\right| \quad + \frac{1}{h_s}\left|\left\{k \in I_s : \bar{d}\left(Y_k, Y_*\right) \geq r_2 + \frac{\varepsilon}{2}\right\}\right|,$$

then, by taking limit $s \to \infty$, from our assumptions, we prove the theorem.



Theorem 3.6. If there exists a sequence $Y=\left(Y_k\right)$ lacunary convergent to X_* and $\overline{d}(X_k,Y_k)\le r$ for each $k\in I_s$, then a sequence $X=\left(X_k\right)$ is $r_\theta-$statistically convergent to X_*.

Proof. From our assumption, we have

$$\lim_s \frac{1}{h_s}\left|\left\{k\in I_s:\overline{d}\left(Y_k,X_*\right)\ge\varepsilon\right\}\right|=0$$

for every $\varepsilon>0$. Because $\overline{d}(X_k,Y_k)\le r$ for each $k\in I_s$, the inclusion

$$\left\{k\in I_s:\overline{d}(Y_k,X_*)\ge\varepsilon\right\}\supseteq\left\{k\in I_s:\overline{d}(X_k,X_*)\ge r+\varepsilon\right\}$$

holds. Hence, we get

$$\lim_s \frac{1}{h_s}\left|\left\{k\in I_s:\overline{d}\left(X_k,X_*\right)\ge r+\varepsilon\right\}\right|=0.$$

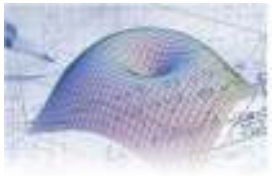
Theorem 3.7. If $X=\left(X_k\right)$ is lacunary statistical convergent to X_*, then

$$S_\theta-LIM^r X_k=\overline{B}_r(X_*):=\left(\mu\in L(\mathbb{R}):\overline{d}(\mu,X_*)\le r\right).$$

Proof. Let $K_1=\left\{k\in I_s:\overline{d}\left(X_k,X_*\right)\ge\varepsilon\right\}$. Since $X_k\xrightarrow{s_\theta}X_*$, we have $\lim_s \frac{1}{h_s}\left|K_1\right|=0$. Let we take $Y\in\overline{B}_r(X_*)$. For all $k\in K_1^{\,c}$, we get

$$\overline{d}(X_k,Y)\le\overline{d}(X_k,X_*)+\overline{d}(X_*,Y)<\varepsilon+r,$$

which shows that $Y\in S_\theta-LIM^r X_k$.



Let $K_2 = \left\{k \in I_s : \overline{d}\left(X_k, X_*\right) \geq r + \varepsilon\right\}$. Now, we take an arbitrary

$Y \in S_\theta - LIM^r X_k$. So, we write $\lim_s \frac{1}{h_s}\left|K_2\right| = 0$. Since $X_k \overset{s_\theta}{\to} X_*$, we have

$\lim_s \frac{1}{h_s}\left|K_1\right| = 0$. For all $k \in K_1^c \bigcap K_2^c$, we get

$$\overline{d}(Y, X_*) \leq \overline{d}(X_k, Y) + \overline{d}(X_k, X_*) < 2\varepsilon + r$$

which shows that $Y \in \overline{B}_r(X_*)$.

Theorem 3.8. For any arbitrary $C \in \Gamma_\theta$ of $X = \left(X_k\right)$, we have $\overline{d}(X_*, C) \leq r$ for all $X_* \in S_\theta - LIM^r X_k$.

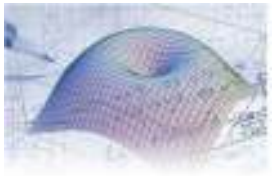
Proof. Assume that there exist a point $C \in \Gamma_\theta$ and $X_* \in S_\theta - LIM^r X_k$ such that $\overline{d}(X_*, C) > r$. Define $\varepsilon = \frac{\overline{d}(X_*, C) - r}{3}$.

Then, we write

$$\left\{k \in I_s : \overline{d}(X_k, X_*) \geq r + \varepsilon\right\} \supseteq \left\{k \in I_s : \overline{d}(X_k, C) < \varepsilon\right\}$$

Since $C \in \Gamma_\theta$, we have $\lim_s \frac{1}{h_s}\left|\left\{k \in I_s : \overline{\mathrm{d}}(\mathrm{X}_k, C) < \varepsilon\right\}\right| \neq 0$.

Hence we get $\lim_s \frac{1}{h_s}\left|\left\{k \in I_s : \overline{d}(X_k, X_*) \geq r + \varepsilon\right\}\right| \neq 0$, which contradicted the fact that $X_* \in S_\theta - LIM^r X_k$.



Theorem 3.9.

a) If $C \in \Gamma_\theta$ of $X = (X_k)$, then $S_\theta - LIM^r X_k \subseteq \overline{B}_r(C)$

b) $S_\theta - LIM^r X_k = \bigcap_{C \in \Gamma_\theta} \overline{B}_r(C) = \left\{ X_* \in L(\mathbb{R}) : \Gamma_\theta \subseteq \overline{B}_r(X_*) \right\}$

Proof.

a) Let $X_* \in S_\theta - LIM^r X_k$ and $C \in \Gamma_\theta$. By Theorem 3.8, we have

$\overline{d}(X_*, C) \le r$ otherwise there exists $\lim_s \frac{1}{h_s} \left| \left\{ k \in I_s : \overline{d}(X_k, X_*) \ge r + \varepsilon \right\} \right| \ne 0$

for $\varepsilon = \frac{\overline{d}(X_*, C) - r}{3}$.

b) By the inclusion in (a), we can write $S_\theta - LIM^r X_k \subseteq \bigcap_{C \in \Gamma_\theta} \overline{B}_r(C)$.

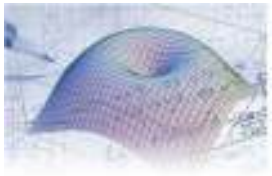
Let $Y \in \bigcap_{C \in \Gamma_\theta} \overline{B}_r(C)$. Then we have $\overline{d}(X_*, C) \le r$ for all $C \in \Gamma_\theta$. This shows that

$\Gamma_\theta \subseteq \overline{B}_r(Y)$, that is , $\bigcap_{C \in \Gamma_\theta} \overline{B}_r(C) = \left\{ X_* \in L(\mathbb{R}) : \Gamma_\theta \subseteq \overline{B}_r(X_*) \right\}$.

Let $Y \notin S_\theta - LIM^r X_k$. Then there exists an $\varepsilon > 0$ such that

$$\lim_s \frac{1}{h_s} \left| \left\{ k \in I_s : \overline{d}(X_k, Y) \ge r + \varepsilon \right\} \right| \ne 0 .$$

This implies the existence of a lacunary statistical cluster point C of X with $\overline{d}(Y, C) > r + \varepsilon$. So, $\Gamma_\theta \not\subset \overline{B}_r(Y)$ and $Y \notin \left\{ X_* \in L(\mathbb{R}) : \Gamma_\theta \subseteq \overline{B}_r(X_*) \right\}$.



Hence, we get $Y \in \left\{ X_* \in L(\mathbb{R}) : \Gamma_\theta \subseteq \overline{B}_r(X_*) \right\}$ which shows that

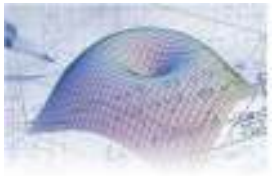
$$\left\{ X_* \in L(\mathbb{R}) : \Gamma_\theta \subseteq \overline{B}_r(X_*) \right\} \subseteq S_\theta - LIM^r X_k.$$

Acknowledgements

The authors are grateful to Professor Salih AYTAR for valuable suggestions and comments.

References

[1] Akçay F. and Aytar S., Rough Convergence of a Sequence of Fuzzy Numbers,Bulletin of Mathematical Analysis and Application, 7(2005), 17-23.
[2] Altın Y., Et M. and Çolak R., Lacunary Statistical and Lacunary Strongly Convergence of Generalized Difference Sequences of fuzzy Numbers, An International Computers and Mathematics With Applications, 52 (2006), 1011-1020.
[3] Aytar S., Rough Statistical Convergence, Numerical Functional Analysis and Optimization, 29(2008), 291-303.
[4] Babaarslan F. and Tuncer A.N., Rough convergence of double sequences of fuzzy numbers, Journal of Applied Analysis and Computation, 10 (2020), 1335-1342.
[5] Das P., Ghosal S. and Ghosh A., Rough statistical convergence of a sequence of random variables in probability, Afrika Matematika, 26 (2015), 1399-1412.
[6] Debnath S. and Rakshit D., Rough Statistical Convergence of a Sequence of Fuzzy Numbers, Mathematica, 61 (2019), 33–39.
[7] Debnath S. and Subramanian N., Generalized rough lacunary statistical triple difference sequence spaces in probability of fractional orde defined by Musielak-Orlicz function, Boletim da Sociedade Paranaense de Matematica, 37 (2019).
[8] Fast, H. (1951). Sur La Convergence Statistique. Colloquium Mathematicae, 2, 241-244.
[9] Fridy, J.A., Orhan, C., Lacunary statistical convergence. Pacific Journal of Mathematics, 160 (1993).
[10] Gökhan A., Et M. and Mursaleen M., Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers, Mathematical and Computer Modelling, 49 (2009), 548-555.
[11] Kişi Ö. and Dündar E., Rough I_2 lacunary statistical convergence of double sequences, Journal of Inequalities and Applications, 230 (2018).



[12] Kwon J. S. and Shim H. T., Remark on Lacunary Statistical Convergence of Fuzzy Numbers, Fuzzy Set and Systems, 123 (2001), 85-88.
[13] Nuray F., Lacunary Statistical Convergence of Sequence of Fuzzy Numbers, Fuzzy Set and Systems, 99 (1998), 353-355.
[14] Phu, H.X.: Rough convergence in normed linear spaces. Numerical Functional Analysis and Optimization 22 (2001), 201-224.
[15] Savaş E., Remark on double lacunary statistical convergence of fuzzy numbers, Journal of Computational Analysis & Applications., 11 (2009), 64-69.