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The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM

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Abstract
In this paper, we first generalize the products of two fuzzy soft matrices. Through these generalizations, three or more fuzzy soft matrices in the different types can be multiplied. Furthermore, we introduce the mean operators and normalized fuzzy weighted mean operators of the fuzzy soft matrices. We discuss the theoretical aspects of these operators. We describe the multicriteria group decision making (MCGDM) problem with different evaluation criterion sets, and then we create two algorithms using the mean operators and generalized products of fuzzy soft matrices to deal with such problems. To show the advantages of the proposed ones, we present the comparison results with some of the preexisting decision making algorithms of fuzzy soft sets. Finally, we create Scilab codes of our algorithms to expedite and facilitate the decision making process.

Keywords Fuzzy soft set · Fuzzy soft matrix · Products of fuzzy soft matrices · Mean operators of fuzzy soft matrices · Decision making

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1 Introduction

Most of the problems encountered in the complexity of the real world involve uncertain or unknown data. One of these problems is also the decision making (DM) problem. The

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attempt to solve DM problems is very important and many mathematical approaches have been proposed for this purpose. In the recent past, many DM methods have been introduced using the approaches such as the fuzzy set, intuitionistic fuzzy set, neutrosophic set and rough set. In addition to these, Molodtsov (1999) defined the soft set which is efficient and practical in the process of DM. Thereafter, many DM procedures based on the soft sets were constructed and their applications in various fields were given. For more details, we refer to read the papers given in (Çağman and Enginoğlu 2010b; Athira et al. 2019; Garg and Arora 2018, 2019; Khan et al. 2019; Feng et al. 2012; Herawan and Deris 2010; Perveen et al. 2019; Selvachandran et al. 2018b; Maji et al. 2002). In Çağman and Enginoğlu (2010a); the authors have defined the soft matrix which is a matrix representation of the soft set and its based approach to solve the DM problems. Immediately following this proposal, several DM algorithms were created by means of the soft matrices which can be summarized in the articles given in (Atagün et al. 2018; Selvachandran et al. 2018a; Basu et al. 2012b; Kamacı 2019; Kamacı et al. 2018a, b, 2019).

Maji et al. (2001) initiated the notion of a fuzzy soft set by embedding the idea of a fuzzy set into the soft set. Cagman et al. (2011) redefined the fuzzy soft sets and then introduced the fuzzy soft aggregation operator which ensures more efficient in the DM processes. Many types of research (Alcantud 2016; Majumdar and Samanta 2010; Xiao et al. 2009) discussed the fuzzy soft set and its some properties. Recently, the fuzzy soft set theory has become important in handling the problems including uncertain data. Yang et al. (2007) advised the parametrization reduction of a fuzzy soft set to simplify the decision phase. Feng et al. (2014) suggested the decomposition of the fuzzy soft sets with finite value spaces to solve the DM problems. Yang et al. (2014) gave a DM method for the quality identification of wines by using fuzzy soft sets. Feng et al. (2010) proposed an adjustable approach to the fuzzy soft set and introduced a weighted fuzzy soft set with applications in DM problems. Roy and Maji (2007) suggested a new fuzzy soft decision making algorithm of object recognition from an uncertain multi observer data. Yang et al. (2013) introduced the multi-fuzzy soft set and its DM model. Gong et al. (2013) introduced the concept of bijective fuzzy soft set, which can be used in DM. Zhan and Zhu (2015) scrutinized the DM methods of fuzzy soft sets and revised some of them. Yang and Ji (2011) described the fuzzy soft matrix representing the fuzzy soft set and constructed a new DM procedure. From a new perspective, Cagman and Enginoglu (2012) redefined the fuzzy soft matrix and reproduced the related operations which make theoretical aspects of the fuzzy soft sets more useful. They presented the products of fuzzy soft matrices and then proposed a fuzzy soft max–min decision making method using these products.

The decision making approaches proposed using the soft sets, fuzzy soft sets and their matrix representations have specific difficulties and limitations. These are sometimes the number of parameters, sometimes the diversity of parameter sets and sometimes the number of decision makers. In this paper, we endeavor to overcome all these difficulties and limitations.

We can present the contributions of this study which aims to propose new concepts and decision making algorithms based on the fuzzy soft matrices as follows:

(1) The fuzzy soft matrix products such as And, And-Not, Or, Or-Not defined by the authors (Cagman and Enginoglu 2012) are generalized so as to multiply more than two fuzzy soft matrices in different types. Thus, these products used in the decision making involving two decision makers can now also be used in group decision making.
(2) It is defined as some mean operators which aggregate fuzzy soft matrices according to a certain order. These mean operators can be used to solve the problems in many practical areas such as group decision making, multicriteria group decision making, and forecasting.

(3) An algorithm (Algorithm 1) which utilizes the mean operators and generalized products is proposed to address the MCGDM with different evaluation criterion sets described in Fan et al. (2018).
(4) It is created an algorithm (Algorithm 2) which achieves more convincing results for the decision making (or MCGDM) problems that can be solved by the previous fuzzy soft decision making algorithms.

This paper is arranged as follows. Section 2 is a brief note on the preliminaries related to the fuzzy soft sets and fuzzy soft matrices. Section 3 is dedicated to the generalizations of some of the fuzzy soft matrix products. Section 4 introduces the mean operators and normalized fuzzy weighted mean operators of fuzzy soft matrices and their relations. In Sect. 5, using new fuzzy soft operations, two fuzzy soft decision making methods are created to solve the MCGDM problems. Also, several examples and comparisons are included to illustrate the practicality and efficiency of these methods. Section 6 presents some conclusions.

2 Preliminaries

This section provides some basic concepts that will form the basis for the next sections.

Molodtsov (1999) defined the concept of a soft set. Later on, Çağman and Enginoğlu (2010b) gave the definition of soft set in the following manner:

Let U be a set of alternatives, $\mathcal{P}(U)$ be the power set of it, X be a criterion set and $A_k \subseteq X$.

Definition 1 (Çağman and Enginoğlu 2010b) A pair (f_{A_k}, X) is said to be a soft set on U iff f_{A_k} is a mapping of X into $\mathcal{P}(U)$. A soft set (f_{A_k}, X) will be denoted as F_{A_k}.

In other words, a soft set F_{A_k} can be written as a set of ordered pairs as follows:

$$F_{A_k} = \{(x, f_{A_k}(x)) : x \in X, f_{A_k}(x) \in \mathcal{P}(U)\}, \tag{1}$$

where $f_{A_k}(x) = \emptyset$ if $x \notin A_k$.

Definition 2 (Zadeh 1965) A fuzzy set τ on U is a set described by a function μ_τ such that μ_τ is a mapping of U into $[0, 1]$. μ_τ is said to be the membership function of τ and also the value $\mu_\tau(u)$ is said to be the membership grade of $u \in U$. This value means the degree of u belonging to fuzzy set τ. Hence, a fuzzy set τ on U can be written following manner:

$$\tau = \{(\mu_\tau(u)/u) : u \in U, \mu_\tau(u) \in [0, 1]\} \tag{2}$$

From now on, $\mathcal{F}(U)$ will denote the set of all fuzzy sets on U. $\Gamma_{A_1}, \Gamma_{A_2}, \Gamma_{A_3}, \ldots$, etc. and $\gamma_{A_1}, \gamma_{A_2}, \gamma_{A_3}, \ldots$, etc. will be used to represent of fuzzy soft sets and their fuzzy approximate functions, respectively.

Definition 3 (Cagman et al. 2011) Let $\gamma_{A_k}(x)$ be a fuzzy set on U for all $x \in X$. Then, a fuzzy soft set Γ_{A_k} on U is a set described by a function γ_{A_k} such that

$$\gamma_{A_k} : X \to \mathcal{F}(U)$$

where $\gamma_{A_k}(x) = \emptyset$ if $x \notin A_k$.

Here, γ_{A_k} is said to be a fuzzy approximate function of the fuzzy soft set Γ_{A_k}, the value $\gamma_{A_k}(x)$ is a fuzzy set which is named as x-element of the fuzzy soft set for all $x \in X$, and $\emptyset$ is the null fuzzy set. Thus, a fuzzy soft set Γ_{A_k} on U can be written as follows:

$$\Gamma_{A_k} = \{(x, \gamma_{A_k}(x)) : x \in X, \gamma_{A_k}(x) \in \mathcal{F}(U)\} \tag{3}$$

Note that from now on, $\mathrm{FS}(U)$ will denote the set of all fuzzy soft sets on U.

Example 1 Suppose that $U = \{u_1, u_2, u_3, u_4\}$ is a set of alternatives and $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is a set of all criteria. If $A_1 = \{x_2, x_3, x_5, x_7\}$ and $\gamma_{A_1} : X \to \mathcal{F}(U)$, $\gamma_{A_1}(x_2) = U$, $\gamma_{A_1}(x_3) = \{0.3/u_1, 0.7/u_2, 1/u_4\}$, $\gamma_{A_1}(x_5) = 0.2/U$, $\gamma_{A_1}(x_7) = \{0.1/u_2\}$, then we write the fuzzy soft set

$$\Gamma_{A_1} = \{(x_2, U), (x_3, \{0.3/u_1, 0.7/u_2, 1/u_4\}), (x_5, 0.2/U), (x_7, \{0.1/u_2\})\}.$$

Definition 4 (Cagman and Enginoglu 2012) Let $\Gamma_{A_1} \in \mathrm{FS}(U)$. Then a fuzzy relation form of Γ_{A_1} is defined as

$$\Re_{A_1} = \{(\mu_{\Re_{A_1}}(u, x)/(u, x)) : (u, x) \in U \times X\}, \tag{4}$$

where the membership function $\mu_{\Re_{A_1}}$ is described by

$$\mu_{\Re_{A_1}} : U \times X \to [0, 1], \mu_{\Re_{A_1}}(u, x) = \mu_{\gamma_{A_1}(x)}(u). \tag{5}$$

Example 2 Consider the fuzzy soft set Γ_{A_1} in Example 1. Then, we obtain a fuzzy relation form of Γ_{A_1} as

$$\Re_{A_1} = \left\{ \begin{array}{l} (1/(u_1, x_2)), (1/(u_2, x_2)), (1/(u_3, x_2)), (1/(u_4, x_2)), \\ (0.3/(u_1, x_3)), (0.7/(u_2, x_3)), (1/(u_4, x_3)), (0.2/(u_1, x_5)), \\ (0.2/(u_2, x_5)), (0.2/(u_3, x_5)), (0.2/(u_4, x_5)), (0.1/(u_2, x_7)) \end{array} \right\}$$

Definition 5 (Cagman and Enginoglu 2012) Let $U = \{u_1, u_2, \ldots, u_t\}$, $X = \{x_1, x_2, \ldots, x_v\}$ and $A_1 \subseteq X$. If Γ_{A_1} is a fuzzy soft set and $\Re_{A_1}$ is a fuzzy relation form of Γ_{A_1}, then the following matrix is called a fuzzy soft matrix representing the fuzzy soft set Γ_{A_1} on U.

$$[a^1_{ij}]_{t\times v} = \begin{bmatrix} a^1_{11} & a^1_{12} & \ldots & a^1_{1v} \\ a^1_{21} & a^1_{22} & \ldots & a^1_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{t1} & a^1_{t2} & \ldots & a^1_{tv} \end{bmatrix} \tag{6}$$

where

$$a^1_{ij} = \begin{cases} \mu_{\gamma_{A_1}(x_j)}(u_i), & \text{if } u_i \in \gamma_{A_1}(x_j) \\ 0, & \text{if } u_i \notin \gamma_{A_1}(x_j) \end{cases}$$

According to the description of $t \times v$ fuzzy soft matrix, a fuzzy soft set Γ_{A_1} is uniquely characterized by the matrix $[a^1_{ij}]$. This implies that a fuzzy soft set Γ_{A_1} is equal to its fuzzy soft matrix $[a^1_{ij}]$ and both of them serve the same purpose.

Notation The set of fuzzy soft matrices of all fuzzy soft sets in $\mathrm{FS}(U)$ will be denoted by $\mathrm{FSM}(U)$. Also, the set of all $t \times v$ fuzzy soft matrices will be symbolized by $\mathrm{FSM}_{t\times v}$.

Example 3 Consider the fuzzy soft set Γ_{A_1} in Example 1 and its fuzzy relation form $\Re_{A_1}$ in Example 2. Then, the fuzzy soft matrix $[a^1_{ij}] \in \mathrm{FSM}_{4\times 7}$ of Γ_{A_1} is

$$[a^1_{ij}]_{4\times 7} = \begin{bmatrix} 0 & 1 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 1 & 0.7 & 0 & 0.2 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0.2 & 0 & 0 \end{bmatrix}$$

Definition 6 Let $[a_{ij}^1]$ be a fuzzy soft matrix and $w_1 \in [0,1]$ be a fuzzy weight for this matrix. Then, $[c_{ij}] = [a_{ij}^1]^{w_1}$ is termed to be a fuzzy-weighted fuzzy soft matrix generated from $[a_{ij}^1]$, where $c_{ij} = w_1 a_{ij}^1$ for all i, j.

Example 4 Consider the fuzzy soft matrix $[a_{ij}^1] \in \mathrm{FSM}_{4\times 7}$ in Example 3. Also, let's take its fuzzy weight as $w_1 = 0.4$. Then, we obtain the fuzzy-weighted fuzzy soft matrix generated from $[a_{ij}^1]$ for $w_1 = 0.4$ as follows:

$$[a_{ij}^1]_{4\times 7}^{w_1} = [a_{ij}^1]_{4\times 7}^{0.4} = \begin{bmatrix} 0 & 0.4 & 0.12 & 0 & 0.08 & 0 & 0 \\ 0 & 0.4 & 0.28 & 0 & 0.08 & 0 & 0.04 \\ 0 & 0.4 & 0 & 0 & 0.08 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0.08 & 0 & 0 \end{bmatrix}$$

Definition 7 (Cagman and Enginoglu 2012) Let $[a_{ij}^1] \in \mathrm{FSM}_{t\times v}$. Then,

(a) $[a_{ij}^1]$ is termed to be a zero fuzzy soft matrix, depicted by [0], if $a_{ij}^1 = 0$ for all i, j.
(b) $[a_{ij}^1]$ is termed to be a universal fuzzy soft matrix, depicted by [1], if $a_{ij}^1 = 1$ for all i, j.

Definition 8 (Cagman and Enginoglu 2012) Let $[a_{ij}^1], [a_{ij}^2] \in \mathrm{FSM}_{t\times v}$. Then,

(a) $[a_{ij}^1]$ is a fuzzy soft submatrix of $[a_{ij}^2]$, depicted by $[a_{ij}^1] \widetilde{\subseteq} [a_{ij}^2]$, if $a_{ij}^1 \le a_{ij}^2$ for all i, j.
(b) $[a_{ij}^1]$ and $[a_{ij}^2]$ are fuzzy soft equal matrices, depicted by $[a_{ij}^1] \widetilde{=} [a_{ij}^2]$, if $a_{ij}^1 = a_{ij}^2$ for all i, j.
(c) $[c_{ij}]$ is called a complement of $[a_{ij}^1]$, depicted by $[c_{ij}] = [a_{ij}^1]^{\circ}$, if $c_{ij} = 1 - a_{ij}^1$ for all i, j.

Lemma 1 *Let $[a_{ij}^1]$ be a fuzzy soft matrix and w_1 be a fuzzy weight of this matrix. Then, we have $[a_{ij}^1]^{w_1} \widetilde{\subseteq} [a_{ij}^1]$.*

Proof It is clear from Definitions 5 and 6 that each fuzzy-weighted fuzzy soft matrix can be considered as a fuzzy soft matrix. Thus, the proof is trivial.

Definition 9 (Cagman and Enginoglu 2012) If $[a_{ij_1}^1], [a_{ij_2}^2] \in \mathrm{FSM}_{t\times v}$, then And product of $[a_{ij_1}^1]$ and $[a_{ij_2}^2]$ is denoted and defined as follows:

$$\wedge : \mathrm{FSM}_{t\times v} \times \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{t\times v^2}, \quad [a_{ij_1}^1] \wedge [a_{ij_2}^2] = [c_{ip}] \tag{7}$$

where $c_{ip} = \min\{a_{ij_1}^1, a_{ij_2}^2\}$ such that $p = v(j_1 - 1) + j_2$.

Definition 10 (Cagman and Enginoglu 2012) If $[a_{ij_1}^1], [a_{ij_2}^2] \in \mathrm{FSM}_{t\times v}$, then Or product of $[a_{ij_1}^1]$ and $[a_{ij_2}^2]$ is denoted and defined as follows:

$$\vee : \mathrm{FSM}_{t\times v} \times \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{t\times v^2}, \quad [a_{ij_1}^1] \vee [a_{ij_2}^2] = [c_{ip}] \tag{8}$$

where $c_{ip} = \max\{a_{ij_1}^1, a_{ij_2}^2\}$ such that $p = v(j_1 - 1) + j_2$.

Definition 11 (Cagman and Enginoglu 2012) If $[a_{ij_1}^1], [a_{ij_2}^2] \in \mathrm{FSM}_{t\times v}$, then And-Not product of $[a_{ij_1}^1]$ and $[a_{ij_2}^2]$ is denoted and defined as follows:

$$\bar{\wedge} : \mathrm{FSM}_{t\times v} \times \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{t\times v^2}, \quad [a_{ij_1}^1] \bar{\wedge} [a_{ij_2}^2] = [c_{ip}] \tag{9}$$

where $c_{ip} = \min\{a_{ij_1}^1, 1 - a_{ij_2}^2\}$ such that $p = v(j_1 - 1) + j_2$.

Definition 12 (Cagman and Enginoglu 2012) If $[a_{ij_1}^1], [a_{ij_2}^2] \in \mathrm{FSM}_{t\times v}$, then Or-Not product of $[a_{ij_1}^1]$ and $[a_{ij_2}^2]$ is denoted and defined as follows:

$$\veebar : \mathrm{FSM}_{t\times v} \times \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{t\times v^2}, \quad [a^1_{ij_1}] \veebar [a^2_{ij_2}] = [c_{ip}] \tag{10}$$

where $c_{ip} = max\{a^1_{ij_1}, 1 - a^2_{ij_2}\}$ such that $p = v(j_1 - 1) + j_2$.

Example 5 Let $U = \{u_1, u_2, u_3\}$ be a set of alternatives and $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of all criteria. Suppose that $A_1 = \{x_2, x_3, x_5\}$ and $A_2 = \{x_1, x_4\}$ are two subsets of the set X. Also, let the fuzzy soft sets

$$\Gamma_{A_1} = \{(x_2, 0.9/U), (x_3, \{0.3/u_1, 0.7/u_2, 0.13/u_3\}), (x_5, U)\},$$
$$\Gamma_{A_2} = \{(x_1, \{0.4/u_1, 0.2/u_2\}), (x_4, \{0.5/u_1, 1/u_2, 1/u_3\})\}.$$

Then, the fuzzy soft matrices of Γ_{A_1} and Γ_{A_2} are respectively as below

$$[a^1_{ij_1}] = \begin{bmatrix} 0 & 0.9 & 0.3 & 0 & 1 \\ 0 & 0.9 & 0.7 & 0 & 1 \\ 0 & 0.9 & 0.13 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [a^2_{ij_2}] = \begin{bmatrix} 0.4 & 0 & 0 & 0.5 & 0 \\ 0.2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, we obtain the fuzzy soft matrix

$$[c_{ip}] = [a^1_{ij_1}] \wedge [a^2_{ij_2}]$$
$$= \begin{bmatrix} 0&0&0&0&0&0.4&0&0&0.5&0&0.3&0&0&0.3&0&0&0&0&0&0&0.4&0&0&0.5&0 \\ 0&0&0&0&0&0.2&0&0&0.9&0&0.2&0&0&0.7&0&0&0&0&0&0&0.2&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0.9&0&0&0&0&0.13&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}.$$

The type of fuzzy soft matrix $[c_{ip}]$ is 3×25.

3 The generalized products of fuzzy soft matrices

Cagman and Enginoglu (2012) defined the And, Or, And-Not, Or-Not products of two fuzzy soft matrices corresponding to two fuzzy soft sets created by using the same criterion set. According to the definition of Çağman and Enginoğlu, to the two fuzzy soft matrices can be multiplied, the numbers of rows and columns of the two fuzzy soft matrices must be mutually equal. In this section, we define the generalized And, Or, And-Not, Or-Not products of two fuzzy soft matrices corresponding two fuzzy soft sets generated by using the different criterion sets. Thus, we can multiply two fuzzy soft matrices, which the numbers of rows are the same but the numbers of columns may be different.

Moreover, while only two fuzzy soft matrices can be multiplied by using the products of Çağman and Enginoğlu, there is no limitation in the number of fuzzy soft matrices for our generalized products.

Let X_1 and X_2 be two criterion sets (where it can be $X_1 \cap X_2 = \emptyset$ or $X_1 \cap X_2 \neq \emptyset$). Also, let $|X_1| = v_1$ and $|X_2| = v_2$ (where $|\cdot|$ is the cardinality of a set).

Let $\Gamma_{X_1}, \Gamma_{X_2} \in \mathrm{FS}(U)$. Also, let $[a^1_{ij_1}]$ and $[a^2_{ij_2}]$ be fuzzy soft matrices corresponding to Γ_{X_1} and Γ_{X_2}, respectively.

Definition 13 The generalized And product of $[a^1_{ij_1}]$ and $[a^2_{ij_2}]$, denoted by $\curlywedge$, is defined by the following

$$\curlywedge : \mathrm{FSM}_{t\times v_1} \times \mathrm{FSM}_{t\times v_2} \longrightarrow \mathrm{FSM}_{t\times v_1 v_2}, \quad [a^1_{ij_1}] \curlywedge [a^2_{ij_2}] = [c_{ip}] \tag{11}$$

where $c_{ip} = \min\{a^1_{ij_1}, a^2_{ij_2}\}$ such that $p = (j_1 - 1)v_2 + j_2$.

Definition 14 The generalized Or product of $[a^1_{ij_1}]$ and $[a^2_{ij_2}]$, denoted by $\curlyvee$, is defined by the following

$$\curlyvee : \mathrm{FSM}_{t\times v_1} \times \mathrm{FSM}_{t\times v_2} \longrightarrow \mathrm{FSM}_{t\times v_1v_2}, \quad [a^1_{ij_1}] \curlyvee [a^2_{ij_2}] = [c_{ip}] \tag{12}$$

where $c_{ip} = \max\{a^1_{ij_1}, a^2_{ij_2}\}$ such that $p = (j_1 - 1)v_2 + j_2$.

Definition 15 The generalized And-Not product of $[a^1_{ij_1}]$ and $[a^2_{ij_2}]$, denoted by $\bar{\curlywedge}$, is defined by the following

$$\bar{\curlywedge} : \mathrm{FSM}_{t\times v_1} \times \mathrm{FSM}_{t\times v_2} \longrightarrow \mathrm{FSM}_{t\times v_1v_2}, \quad [a^1_{ij_1}] \bar{\curlywedge} [a^2_{ij_2}] = [c_{ip}] \tag{13}$$

where $c_{ip} = \min\{a^1_{ij_1}, 1 - a^2_{ij_2}\}$ such that $p = (j_1 - 1)v_2 + j_2$.

Definition 16 The generalized Or-Not product of $[a^1_{ij_1}]$ and $[a^2_{ij_2}]$, denoted by $\underline{\curlyvee}$, is defined by the following

$$\underline{\curlyvee} : \mathrm{FSM}_{t\times v_1} \times \mathrm{FSM}_{t\times v_2} \longrightarrow \mathrm{FSM}_{t\times v_1v_2}, \quad [a^1_{ij_1}] \underline{\curlyvee} [a^2_{ij_2}] = [c_{ip}] \tag{14}$$

where $c_{ip} = \max\{a^1_{ij_1}, 1 - a^2_{ij_2}\}$ such that $p = (j_1 - 1)v_2 + j_2$.

It is remarked that the type of fuzzy soft matrix $[c_{ip}]$ in these definitions is $t \times v_1v_2$.

Example 6 Let $U = \{u_1, u_2, u_3\}$ be a set of alternatives. Also, let $X_1 = \{x^1_1, x^1_2, x^1_3\}$ and $X_2 = \{x^2_1, x^2_2\}$ be two criterion sets, where X_1, X_2 denote the main criteria and x^1, x^2 denote the sub-criteria related to X_1, X_2 respectively. Assume that the fuzzy soft sets

$$\Gamma_{X_1} = \{(x^1_1, 0.9/U), (x^1_2, \{0.3/u_1, 0.7/u_2, 0.13/u_3\}), (x^1_3, U)\},$$
$$\Gamma_{X_2} = \{(x^2_1, \{0.4/u_1, 0.2/u_2\}), (x^2_2, \{0.5/u_1, 1/u_2, 1/u_3\})\}.$$

Then, we obtain the following fuzzy soft matrices

$$[a^1_{ij_1}] = \begin{bmatrix} 0.9 & 0.3 & 1 \\ 0.9 & 0.7 & 1 \\ 0.9 & 0.13 & 1 \end{bmatrix}; \quad \textit{and} \quad [a^2_{ij_2}] = \begin{bmatrix} 0.4 & 0.5 \\ 0.2 & 1 \\ 0 & 1 \end{bmatrix}$$

The generalized And product of $[a^1_{ij_1}] \in \mathrm{FSM}_{3\times 3}$ and $[a^2_{ij_2}] \in \mathrm{FSM}_{3\times 2}$ is

$$[c_{ip}] = [a^1_{ij_1}] \curlywedge [a^2_{ij_2}] = \begin{bmatrix} 0.4 & 0.5 & 0.3 & 0.3 & 0.4 & 0.5 \\ 0.2 & 0.9 & 0.2 & 0.7 & 0.2 & 1 \\ 0 & 0.9 & 0 & 0.13 & 0 & 1 \end{bmatrix}.$$

The type of fuzzy soft matrix $[c_{ip}]$ is 3×6.

Proposition 1 *Let $[a^1_{ij_1}] \in FSM_{t\times v_1}$, $[a^2_{ij_2}] \in FSM_{t\times v_2}$ be two fuzzy soft matrices. Then, generalized products hold the following De Morgan's Laws:*

(i) $([a^1_{ij_1}] \curlyvee [a^2_{ij_2}])^\circ = [a^1_{ij_1}]^\circ \curlywedge [a^2_{ij_2}]^\circ$.
(ii) $([a^1_{ij_1}] \curlywedge [a^2_{ij_2}])^\circ = [a^1_{ij_1}]^\circ \curlyvee [a^2_{ij_2}]^\circ$.
(iii) $([a^1_{ij_1}] \underline{\curlyvee} [a^2_{ij_2}])^\circ = [a^1_{ij_1}]^\circ \bar{\curlywedge} [a^2_{ij_2}]^\circ$.
(iv) $([a^1_{ij_1}] \bar{\curlywedge} [a^2_{ij_2}])^\circ = [a^1_{ij_1}]^\circ \underline{\curlyvee} [a^2_{ij_2}]^\circ$.

Proof (i) Let $[a^1_{ij_1}] \in \mathrm{FSM}_{t\times v_1}$, $[a^2_{ij_2}] \in \mathrm{FSM}_{t\times v_2}$ and also $[c_{ip}] = ([a^1_{ij_1}] \curlyvee [a^2_{ij_2}])^\circ$, $[d_{ir}] = [a^1_{ij_1}]^\circ \curlywedge [a^2_{ij_2}]^\circ$. Since the types of $[c_{ip}]$ and $[d_{ir}]$ are $t \times v_1v_2$ and $c_{ip} =$

$1 - \max\{a^1_{ij_1}, a^2_{ij_2}\} = \min\{1 - a^1_{ij_1}, 1 - a^2_{ij_2}\} = d_{ir}$ for all i, j_1, j_2, we have $[c_{ip}] = [d_{ir}]$. This completes the proof of part (i).

Likewise, the parts (ii)–(iv) can be illustrated.

Theorem 1 *The products newly-introduced in Definitions 13, 14, 15 and 16 are generalizations of the products introduced in Definitions 9, 10, 11 and 12, respectively.*

Proof Let $[a^1_{ij_1}] \in \mathrm{FSM}_{t\times v_1}$, $[a^2_{ij_2}] \in \mathrm{FSM}_{t\times v_2}$ be two fuzzy soft matrices. Let's consider the generalized And product. If $v_1 = v_2 = v$, then we obtain the type of fuzzy soft matrix $[c_{ip}] = [a^1_{ij_1}] \curlywedge [a^2_{ij_2}]$ is $t \times v^2$ and $p = (j_1 - 1)v + j_2$ by Definition 13. Also, we obtain $[c_{ip}] = [a^1_{ij_1}] \wedge [a^2_{ij_2}]$ and $p = v(j_1 - 1) + j_2$ by using Or product in Definition 9. Therefore, Definition 13 is a generalization of Definition 9. It can be proved similarly to other products.

Theorem 2 *The generalized And product is associative, i.e. if* $[a^1_{ij_1}] \in FSM_{t\times v_1}$, $[a^2_{ij_2}] \in FSM_{t\times v_2}$ *and* $[a^3_{ij_3}] \in FSM_{t\times v_3}$ *then* $([a^1_{ij_1}] \curlywedge [a^2_{ij_2}]) \curlywedge [a^3_{ij_3}] = [a^1_{ij_1}] \curlywedge ([a^2_{ij_2}] \curlywedge [a^3_{ij_3}])$.

Proof Let $[a^1_{ij_1}] \in SM_{t\times v_1}$, $[a^2_{ij_2}] \in SM_{t\times v_2}$ and $[a^3_{ij_3}] \in \mathrm{FSM}_{t\times v_3}$. By Definition 13, we can write $[a^1_{ij_1}] \curlywedge [a^2_{ij_2}] = [c_{ip}] \in \mathrm{FSM}_{t\times v_1 v_2}$ where $c_{ip} = \min\{a^1_{ij_1}, a^2_{ij_2}\}$ such that $p = (j_1 - 1)v_2 + j_2$. Likewise, we can write $[c_{ip}] \curlywedge [a^3_{ij_3}] = [d_{ir}] \in \mathrm{FSM}_{t\times v_1 v_2 v_3}$ where $d_{ir} = \min\{c_{ip}, a^3_{ij_3}\}$ such that $r = (p - 1)v_3 + j_3$. Then, we obtain

$$\begin{aligned} r &= (p-1)v_3 + j_3 \\ &= ((j_1 - 1)v_2 + j_2 - 1)v_3 + j_3 \\ &= (j_1 - 1)v_2 v_3 + (j_2 - 1)v_3 + j_3 \end{aligned} \tag{15}$$

Similarly, we can write $[a^2_{ij_2}] \curlywedge [a^3_{ij_3}] = [f_{iq}] \in \mathrm{FSM}_{t\times v_2 v_3}$ where $f_{iq} = \min\{a^2_{ij_2}, a^3_{ij_3}\}$ such that $q = (j_2 - 1)v_3 + j_3$. Likewise, we can write $[a^1_{ij_1}] \curlywedge [f_{iq}] = [g_{is}] \in \mathrm{FSM}_{t\times v_1 v_2 v_3}$ where $g_{is} = \min\{a^1_{ij_1}, f_{iq}\}$ such that $s = (j_1 - 1)v_2 v_3 + q$. Then, we obtain

$$\begin{aligned} s &= (j_1 - 1)v_2 v_3 + q \\ &= (j_1 - 1)v_2 v_3 + (j_2 - 1)v_3 + j_3 \end{aligned} \tag{16}$$

Since $\min\{\min\{a^1_{ij_1}, a^2_{ij_2}\}, a^3_{ij_3}\} = \min\{a^1_{ij_1}, \min\{a^2_{ij_2}, a^3_{ij_3}\}\}$ and also by Eqs. (15) and (16), it is clear that $[d_{ir}] = [g_{is}]$. Thus, the proof is completed.

Theorem 3 *The generalized Or product is associative, i.e. if* $[a_{ij}] \in FSM_{t\times v_1}$, $[b_{ik}] \in FSM_{t\times v_2}$ *and* $[c_{il}] \in FSM_{t\times v_3}$, *then* $([a_{ij}] \curlyvee [b_{ik}]) \curlyvee [c_{il}] = [a_{ij}] \curlyvee ([b_{ik}] \curlyvee [c_{il}])$.

Proof It is similar to the proof of Theorem 2.

The generalized Or-Not product and And-Not product are not associative, an implementation of this is below.

Example 7 Suppose that $[a^1_{ij_1}] \in \mathrm{FSM}_{3\times 3}$, $[a^2_{ij_2}] \in \mathrm{FSM}_{3\times 2}$ and $[a^3_{ij_3}] \in \mathrm{FSM}_{3\times 1}$ are given as follows:

$$[a^1_{ij_1}] = \begin{bmatrix} 0 & 0.1 \\ 1 & 0 \\ 0.9 & 0 \end{bmatrix}, \quad [a^2_{ij_2}] = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} \quad \text{and}$$

$$[a^3_{ij_3}] = \begin{bmatrix} 0 \\ 0 \\ 0.4 \end{bmatrix}.$$

Then, it is seen that $([a^1_{ij_1}] \underline{\curlyvee} [a^2_{ij_2}]) \underline{\curlyvee} [a^3_{ij_3}] \neq [a^1_{ij_1}] \underline{\curlyvee} ([a^2_{ij_2}] \underline{\curlyvee} [a^3_{ij_3}])$ since

$$([a^1_{ij_1}] \underline{\curlyvee} [a^2_{ij_2}]) \underline{\curlyvee} [a^3_{ij_3}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0.9 & 1 & 1 & 0.6 \end{bmatrix}$$

and

$$[a^1_{ij_1}] \underline{\curlyvee} ([a^2_{ij_2}] \underline{\curlyvee} [a^3_{ij_3}]) = \begin{bmatrix} 0 & 0 & 0 & 0.1 & 0.1 & 0.1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0.9 & 0.9 & 0.9 & 0.4 & 0.4 & 0.1 \end{bmatrix}.$$

Similarly, it is seen that $([a^1_{ij_1}] \overline{\curlywedge} [a^2_{ij_2}]) \overline{\curlywedge} [a^3_{ij_3}] \neq [a^1_{ij_1}] \overline{\curlywedge} ([a^2_{ij_2}] \overline{\curlywedge} [a^3_{ij_3}])$ since

$$([a^1_{ij_1}] \overline{\curlywedge} [a^2_{ij_2}]) \overline{\curlywedge} [a^3_{ij_3}] = \begin{bmatrix} 0 & 0 & 0 & 0.1 & 0.1 & 0.1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0.6 & 0.6 & 0.1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[a^1_{ij_1}] \overline{\curlywedge} ([a^2_{ij_2}] \overline{\curlywedge} [a^3_{ij_3}]) = \begin{bmatrix} 0 & 0 & 0 & 0.1 & 0.1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 2 *For FSM(U), we have*

(i) $[1]_{t\times 1}$ *is a unit element of FSM(U) for the generalized And product.*
(ii) $[0]_{t\times 1}$ *is a unit element of FSM(U) for the generalized Or product.*

Proof (i) Let $[a^1_{ij_1}] \in \mathrm{FSM}_{t\times v_1}$ be a fuzzy soft matrix and also $[c_{ip}] = [a^1_{ij_1}] \curlywedge [1]_{t\times 1}$. Since the type of fuzzy soft matrix $[c_{ip}]$ is $t \times v_1$ and $c_{ip} = \min\{a^1_{ij_1}, 1\} = a^1_{ij_1}$, it is obtained $[c_{ip}] = [a^1_{ij_1}]$ by Definition 13.
(ii) It can be demonstrated in a similar way.

Theorem 4 *For the generalized And product and generalized Or product, the following results holds*

(i) *Considering the generalized And product, FSM(U) is a monoid.*
(ii) *Considering the generalized Or product, FSM(U) is a monoid.*

Proof It is trivial by Theorems 2, 3 and Proposition 2.

4 Some special operators of fuzzy soft matrices

In 1995, Chang and Lee (1995) put forward that the prediction of normalized set of fuzzy weights are one of the most important issues in the fuzzy multi-attribute decision making process. Following this opinion, they were the first to define the concept of fuzzy normalization. Subsequently, many studies discussing the normalized fuzzy weights and their applications in the decision making were published (Elevli 2014; Guh et al. 2008; Wang and Elhag 2006). In this section, we first introduce arithmetic mean, geometric mean, harmonic mean and quadratic mean operators of the fuzzy soft matrices. Later, by means of the notion of fuzzy normalization, the normalized fuzzy weighted mean operators of the fuzzy soft matrices are generated.

Definition 17 (*Arithmetic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called an arithmetic mean (AM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_1 [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_1 [a_{ij}^2] \bigoplus_1 \cdots \bigoplus_1 [a_{ij}^s]$, if for all i, j

$$c_{ij} = \frac{1}{s}\sum_{\ell=1}^{s} a_{ij}^\ell. \tag{17}$$

Definition 18 (*Normalized fuzzy weighted arithmetic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a normalized fuzzy weighted arithmetic mean (NFWAM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{1_{\hat{w}}} [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{1_{\hat{w}}} [a_{ij}^2] \bigoplus_{1_{\hat{w}}} \cdots \bigoplus_{1_{\hat{w}}} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \sum_{\ell=1}^{s} \hat{w}_\ell a_{ij}^\ell, \tag{18}$$

where $\hat{w}_\ell = w_\ell/(w_1 + w_2 + \cdots + w_s)$ is referred to as a normalized fuzzy weight for the fuzzy soft matrix $[a_{ij}^\ell]$. Also, $w_1, w_2, \ldots, w_s \in [0, 1]$ respectively indicate the fuzzy weights for the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$ such that $w_1 + w_2 + \cdots + w_s > 0$.

Definition 19 (*Geometric mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a geometric mean (GM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_0 [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_0 [a_{ij}^2] \bigoplus_0 \ldots \bigoplus_0 [a_{ij}^s]$, if for all i, j

$$c_{ij} = \left(\prod_{\ell=1}^{s} a_{ij}^\ell\right)^{\frac{1}{s}}. \tag{19}$$

Definition 20 (*Normalized fuzzy weighted geometric mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a normalized fuzzy weighted geometric mean (NFWGM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{0_{\hat{w}}} [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{0_{\hat{w}}} [a_{ij}^2] \bigoplus_{0_{\hat{w}}} \cdots \bigoplus_{0_{\hat{w}}} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \prod_{\ell=1}^{s} (a_{ij}^\ell)^{\hat{w}_\ell}, \tag{20}$$

where $\hat{w}_\ell = w_\ell/(w_1 + w_2 + \cdots + w_s)$ is referred to as a normalized fuzzy weight for the fuzzy soft matrix $[a_{ij}^\ell]$. Also, $w_1, w_2, \ldots, w_s \in [0, 1]$ respectively indicate the fuzzy weights for the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$ such that $w_1 + w_2 + \cdots + w_s > 0$.

Definition 21 (*Harmonic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$ such that $a_{ij}^\ell \neq 0$ for all $\ell = 1, 2, \ldots, s$. Then $[c_{ij}]$ is called a harmonic mean (HM) of

$[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{-1} [a_{ij}^2] \bigoplus_{-1} \ldots \bigoplus_{-1} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \left(\frac{1}{s} \sum_{\ell=1}^{s} \frac{1}{a_{ij}^\ell} \right)^{-1}. \tag{21}$$

Definition 22 (*Normalized fuzzy weighted harmonic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$ such that $a_{ij}^\ell \neq 0$ for all $\ell = 1, 2, \ldots, s$. Then $[c_{ij}]$ is called a normalized fuzzy weighted harmonic mean (NFWHM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{-1_{\hat{w}}}[a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{-1_{\hat{w}}} [a_{ij}^2] \bigoplus_{-1_{\hat{w}}} \ldots \bigoplus_{-1_{\hat{w}}} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \left(\sum_{\ell=1}^{s} \frac{\hat{w}_\ell}{a_{ij}^\ell} \right)^{-1}, \tag{22}$$

where $\hat{w}_\ell = w_\ell/(w_1 + w_2 + \cdots + w_s)$ is referred to as a normalized fuzzy weight for the fuzzy soft matrix $[a_{ij}^\ell]$. Also, $w_1, w_2, \ldots, w_s \in [0, 1]$ respectively indicate the fuzzy weights for the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$ such that $w_1 + w_2 + \cdots + w_s > 0$.

Definition 23 (*Quadratic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a quadratic mean (QM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{2}[a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{2} [a_{ij}^2] \bigoplus_{2} \ldots \bigoplus_{2} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \left(\frac{1}{s} \sum_{\ell=1}^{s} (a_{ij}^\ell)^2 \right)^{\frac{1}{2}}. \tag{23}$$

Definition 24 (*Normalized fuzzy weighted quadratic mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a normalized fuzzy weighted quadratic mean (NFWQM) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{2_{\hat{w}}}[a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{2_{\hat{w}}} [a_{ij}^2] \bigoplus_{2_{\hat{w}}} \ldots \bigoplus_{2_{\hat{w}}} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \left(\sum_{\ell=1}^{s} \hat{w}_\ell (a_{ij}^\ell)^2 \right)^{\frac{1}{2}}, \tag{24}$$

where $\hat{w}_\ell = w_\ell/(w_1 + w_2 + \cdots + w_s)$ is referred to as a normalized fuzzy weight for the fuzzy soft matrix $[a_{ij}^\ell]$. Also, $w_1, w_2, \ldots, w_s \in [0, 1]$ respectively indicate the fuzzy weights for the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$ such that $w_1 + w_2 + \cdots + w_s > 0$.

Example 8 Let $U = \{u_1, u_2, u_3, u_4\}$ be a set of alternatives and $X_1 = \{x_1^1, x_2^1, x_3^1\}$ be a criterion set. Also, let's take the fuzzy soft sets

$$\Gamma_{X_1}^1 = \left\{ \begin{array}{l} (x_1^1, \{0.1/u_1, 0.2/u_2, 0.1/u_3, 0.5/u_4\}), \\ (x_2^1, \{0.1/u_1, 1/u_2, 0.3/u_3, 0.1/u_4\}), \\ (x_3^1, 0.4/U) \end{array} \right\},$$

$$\Gamma_{X_1}^2 = \left\{ \begin{array}{l} (x_1^1, \{0.9/u_1, 0.2/u_2, 0.1/u_3, 0.5/u_4\}), \\ (x_2^1, \{0.1/u_1, 1/u_2, 0.3/u_3, 0.4/u_4\}), \\ (x_3^1, 0.1/U) \end{array} \right\},$$

$$\Gamma_{X_1}^3 = \left\{ \begin{array}{l} (x_1^1, \{0.2/u_1, 1/u_2, 0.1/u_3, 0.2/u_4\}), \\ (x_2^1, U), \\ (x_3^1, \{0.4/u_1, 1/u_2, 0.1/u_3, 0.7/u_4\}) \end{array} \right\}.$$

We write the fuzzy soft matrices of $\Gamma_{X_1}^1$, $\Gamma_{X_1}^2$ and $\Gamma_{X_1}^3$ as follows:

$$[a_{ij_1}^1] = \begin{bmatrix} 0.1 & 0.1 & 0.4 \\ 0.2 & 1 & 0.4 \\ 0.1 & 0.3 & 0.4 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}, [a_{ij_2}^2] = \begin{bmatrix} 0.9 & 0.1 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.5 & 0.4 & 0.1 \end{bmatrix} \text{ and } [a_{ij_3}^3] = \begin{bmatrix} 0.2 & 1 & 0.4 \\ 1 & 1 & 1 \\ 0.1 & 1 & 0.1 \\ 0.2 & 1 & 0.7 \end{bmatrix}$$

Then, we obtain

$$[c_{ij}] = \bigoplus_{\ell=1}^{3}{}_1[a_{ij}^\ell] = \begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.466 & 1 & 0.5 \\ 0.1 & 0.533 & 0.2 \\ 0.4 & 0.5 & 0.4 \end{bmatrix}, [d_{ij}] = \bigoplus_{\ell=1}^{2}{}_0[a_{ij}^\ell] = \begin{bmatrix} 0.3 & 0.1 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.1 & 0.3 & 0.2 \\ 0.5 & 0.2 & 0.2 \end{bmatrix},$$

$$[e_{ij}] = \bigoplus_{\ell=2}^{3}{}_{-1}[a_{ij}^\ell] = \begin{bmatrix} 0.327 & 0.181 & 0.16 \\ 0.333 & 1 & 0.181 \\ 0.1 & 0.461 & 0.1 \\ 0.285 & 0.571 & 0.175 \end{bmatrix} \text{ and}$$

$$[f_{ij}] = \bigoplus_{\ell=1}^{2}{}_2[a_{ij}^\ell] = \begin{bmatrix} 0.64 & 0.1 & 0.291 \\ 0.2 & 1 & 0.291 \\ 0.1 & 0.3 & 0.291 \\ 0.5 & 0.291 & 0.291 \end{bmatrix}.$$

Assume that $w = (w_1, w_2, w_3) = (0.8, 0.5, 0.7)$. Then, we have the following fuzzy soft matrices:

$$[g_{ij}] = \bigoplus_{\ell=1}^{3}{}_{1_{\hat{w}}}[a_{ij}^\ell] = \begin{bmatrix} 0.335 & 0.415 & 0.325 \\ 0.48 & 1 & 0.535 \\ 0.1 & 0.545 & 0.22 \\ 0.395 & 0.49 & 0.43 \end{bmatrix} \text{ and}$$

$$[h_{ij}] = \bigoplus_{\ell=1}^{2}{}_{-1_{\hat{w}}}[a_{ij}^\ell] = \begin{bmatrix} 0.152 & 0.1 & 0.185 \\ 0.2 & 1 & 0.185 \\ 0.1 & 0.3 & 0.185 \\ 0.5 & 0.14 & 0.185 \end{bmatrix}.$$

Proposition 3 *Let* $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in FSM_{t\times v}$. *Then,*

(i) $\bigoplus_{\ell=1}^{s}{}_{0}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{2}[a_{ij}^{\ell}]$.

(ii) $\bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{\kappa}[a_{ij}^{\ell}]$ *for* $\kappa = 0, 1, 2$ *if* $a_{ij}^{\ell} \neq 0$ *for all* $\ell = 1, 2, \ldots, s$.

Proof (i) Let $[a_{ij}^{1}], [a_{ij}^{2}], \ldots, [a_{ij}^{s}] \in \mathrm{FSM}_{t\times v}$.

By Cauchy–Schwarz inequality, we can write for all i and j

$$\begin{aligned} &\sum_{\ell=1}^{s}\left(a_{ij}^{\ell}\right)^{2}\sum_{\ell=1}^{s}1 \geq \left(\sum_{\ell=1}^{s}a_{ij}^{\ell}\right)^{2} \\ &\Rightarrow \frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{2} \geq \left(\frac{1}{s}\sum_{\ell=1}^{s}a_{ij}^{\ell}\right)^{2} \\ &\Rightarrow \left(\frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{2}\right)^{\frac{1}{2}} \geq \frac{1}{s}\sum_{\ell=1}^{s}a_{ij}^{\ell}. \end{aligned} \tag{25}$$

Thus, we prove that $\bigoplus_{\ell=1}^{s}{}_{1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{2}[a_{ij}^{\ell}]$.

Since the logarithm function is concave, we have for all i and j

$$\log\left(\frac{1}{s}\sum_{\ell=1}^{s}a_{ij}^{\ell}\right) \geq \frac{1}{s}\sum_{\ell=1}^{s}\log(a_{ij}^{\ell}) = \frac{1}{s}\log\left(\prod_{\ell=1}^{s}a_{ij}^{\ell}\right) = \log\left(\prod_{\ell=1}^{s}a_{ij}^{\ell}\right)^{\frac{1}{s}}. \tag{26}$$

Then, we write that

$$\frac{1}{s}\sum_{\ell=1}^{s}a_{ij}^{\ell} \geq \left(\prod_{\ell=1}^{s}a_{ij}^{\ell}\right)^{\frac{1}{s}}. \tag{27}$$

Thus, we prove that $\bigoplus_{\ell=1}^{s}{}_{0}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{1}[a_{ij}^{\ell}]$.

(ii) Let $[a_{ij}^{1}], [a_{ij}^{2}], \ldots, [a_{ij}^{s}] \in \mathrm{FSM}_{t\times v}$ such that $a_{ij}^{\ell} \neq 0$ for all $\ell = 1, 2, \ldots, s$.
Then, by case (i), we have

$$\bigoplus_{\ell=1}^{s}{}_{0}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{2}[a_{ij}^{\ell}].$$

Now, let's prove that $\bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{0}[a_{ij}^{\ell}]$.

By Eq. (27), we can write for all i and j

$$\frac{1}{s}\sum_{\ell=1}^{s}\frac{1}{a_{ij}^{\ell}} \geq \left(\prod_{\ell=1}^{s}\frac{1}{a_{ij}^{\ell}}\right)^{\frac{1}{s}} \Rightarrow \frac{s}{\sum_{\ell=1}^{s}\frac{1}{a_{ij}^{\ell}}} \leq \left(\prod_{\ell=1}^{s}a_{ij}^{\ell}\right)^{\frac{1}{s}}. \tag{28}$$

Thus, we have that $\bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{0}[a_{ij}^{\ell}]$.

Proposition 4 *Let* $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in FSM_{t\times v}$. *Then*

(i) $\bigoplus_{\ell=1}^{s} {}_{0_{\hat{w}}}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s} {}_{1_{\hat{w}}}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s} {}_{2_{\hat{w}}}[a_{ij}^{\ell}]$.

(ii) $\bigoplus_{\ell=1}^{s} {}_{-1_{\hat{w}}}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s} {}_{\kappa_{\hat{w}}}[a_{ij}^{\ell}]$ *for* $\kappa = 0, 1, 2$ *if* $a_{ij}^{\ell} \neq 0$ *for all* $\ell = 1, 2, \ldots, s$.

Proof The proofs can be proved by similar techniques used in the proofs of Proposition 3. □

Example 9 Let us take the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2] \in \mathrm{FSM}_{4\times 3}$ given in Example 8. Also, we consider $[d_{ij}] = \bigoplus_{\ell=1}^{2} {}_{0}[a_{ij}^{\ell}]$ and $[f_{ij}] = \bigoplus_{\ell=1}^{2} {}_{2}[a_{ij}^{\ell}]$ in Example 8. In addition to these fuzzy soft matrices, we find that

$$\bigoplus_{\ell=1}^{2} {}_{1}[a_{ij}^{\ell}] = \begin{bmatrix} 0.5 & 0.1 & 0.25 \\ 0.2 & 1 & 0.25 \\ 0.1 & 0.3 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{bmatrix} \text{ and } \bigoplus_{\ell=1}^{2} {}_{-1}[a_{ij}^{\ell}] = \begin{bmatrix} 0.18 & 0.1 & 0.16 \\ 0.2 & 1 & 0.16 \\ 0.1 & 0.3 & 0.16 \\ 0.5 & 0.16 & 0.16 \end{bmatrix}.$$

From Definition 8 (a), we say that $\bigoplus_{\ell=1}^{2} {}_{-1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{2} {}_{0}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{2} {}_{1}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus {}_{2_{\ell=1}}^{2}[a_{ij}^{\ell}]$

Proposition 5 *Let* $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in FSM_{t\times v}$. *Then*

(i) *If* $[a_{ij}^1] = [a_{ij}^2] = \cdots = [a_{ij}^s]$ *then* $\bigoplus_{\ell=1}^{s} {}_{0_{\hat{w}}}[a_{ij}^{\ell}] = \bigoplus_{\ell=1}^{s} {}_{1_{\hat{w}}}[a_{ij}^{\ell}] = \bigoplus_{\ell=1}^{s} {}_{2_{\hat{w}}}[a_{ij}^{\ell}]$.

(ii) *If* $[a_{ij}^1] = [a_{ij}^2] = \cdots = [a_{ij}^s]$ *and* $a_{ij}^{\ell} \neq 0$ *for all* $\ell = 1, 2, \ldots, s$ *then for* $\kappa = 0, 1, 2$, *we have*

$$\bigoplus_{\ell=1}^{s} {}_{-1_{\hat{w}}}[a_{ij}^{\ell}] = \bigoplus_{\ell=1}^{s} {}_{\kappa_{\hat{w}}}[a_{ij}^{\ell}]$$

Proof (i) Let $[a_{ij}^{\ell}] = [a_{ij}]$ for all $\ell = 1, 2, \ldots, s$. Then, we have for all i and j

$$\begin{aligned} \prod_{\ell=1}^{s} (a_{ij}^{\ell})^{\hat{w}_{\ell}} &= (a_{ij}^1)^{\hat{w}_1} (a_{ij}^2)^{\hat{w}_2} \ldots (a_{ij}^s)^{\hat{w}_s} \\ &= (a_{ij})^{\frac{w_1}{\sum_{\ell=1}^{s} w_{\ell}}} (a_{ij})^{\frac{w_2}{\sum_{\ell=1}^{s} w_{\ell}}} \ldots (a_{ij})^{\frac{w_s}{\sum_{\ell=1}^{s} w_{\ell}}} \\ &= a_{ij} \end{aligned}$$

$$\begin{aligned} \sum_{\ell=1}^{s} \hat{w}_{\ell} a_{ij}^{\ell} &= \hat{w}_1 a_{ij}^1 + \hat{w}_2 a_{ij}^2 + \cdots + \hat{w}_s a_{ij}^s \\ &= \frac{w_1}{\sum_{\ell=1}^{s} w_{\ell}} a_{ij} + \frac{w_2}{\sum_{\ell=1}^{s} w_{\ell}} a_{ij} + \cdots + \frac{w_s}{\sum_{\ell=1}^{s} w_{\ell}} a_{ij} \\ &= a_{ij} \end{aligned}$$

and

$$\left(\sum_{\ell=1}^{s}\hat{w}_\ell(a_{ij}^\ell)^2\right)^{\frac{1}{2}} = \left(\hat{w}_1(a_{ij}^1)^2+\hat{w}_2(a_{ij}^2)^2+\cdots+\hat{w}_s(a_{ij}^s)^2\right)^{\frac{1}{2}}$$
$$= \left(\frac{w_1}{\sum_{\ell=1}^{s}w_\ell}(a_{ij})^2+\frac{w_2}{\sum_{\ell=1}^{s}w_\ell}(a_{ij})^2+\cdots+\frac{w_s}{\sum_{\ell=1}^{s}w_\ell}(a_{ij})^2\right)^{\frac{1}{2}}$$
$$= a_{ij}$$

Hence, we get $\bigoplus_{\ell=1}^{s}{}_0[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_1[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_2[a_{ij}^\ell]$.

(ii) Let $[a_{ij}^\ell] = [a_{ij}]$ for all $\ell = 1, 2, \ldots, s$ and $a_{ij} \neq 0$ for all i, j. Then, we obtain for all i and j

$$\left(\sum_{\ell=1}^{s}\frac{\hat{w}_\ell}{a_{ij}^\ell}\right)^{-1} = \left(\frac{\hat{w}_1}{a_{ij}^1}+\frac{\hat{w}_2}{a_{ij}^2}+\cdots+\frac{\hat{w}_s}{a_{ij}^s}\right)^{-1}$$
$$= \left(\frac{w_1}{\sum_{\ell=1}^{s}w_\ell}\frac{1}{a_{ij}}+\frac{w_2}{\sum_{\ell=1}^{s}w_\ell}\frac{1}{a_{ij}}+\cdots+\frac{w_s}{\sum_{\ell=1}^{s}w_\ell}\frac{1}{a_{ij}}\right)^{-1}$$
$$= a_{ij}$$

In addition to that, considering the equalities in part (i), we have $\bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_\kappa[a_{ij}^\ell]$ for $\kappa = 0, 1, 2$.

Proposition 6 *Let* $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in FSM_{t\times v}$.

(i) *If* $[a_{ij}^1] = [a_{ij}^2] = \cdots = [a_{ij}^s]$ *then* $\bigoplus_{\ell=1}^{s}{}_0[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_1[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_2[a_{ij}^\ell]$.

(ii) *If* $[a_{ij}^1] = [a_{ij}^2] = \cdots = [a_{ij}^s]$ *and* $a_{ij}^\ell \neq 0$ *for all* $\ell = 1, 2, \ldots, s$ *then for* $\kappa = 0, 1, 2$, *we have*

$$\bigoplus_{\ell=1}^{s}{}_{-1}[a_{ij}^\ell] = \bigoplus_{\ell=1}^{s}{}_\kappa[a_{ij}^\ell]$$

Proof The proofs can be demonstrated by similar techniques used in the proofs of Proposition 5.

Proposition 7 *Let* $[a_{ij}^1], [a_{ij}^2] \in FSM_{t\times v}$. *Then, the mean operators are commutative, i.e.* $[a_{ij}^1]\star[a_{ij}^2] = [a_{ij}^2]\star[a_{ij}^1]$ *for each* $\star \in \{\bigoplus_1, \bigoplus_0, \bigoplus_{-1}, \bigoplus_2, \bigoplus_{1_{\hat{w}}}, \bigoplus_{0_{\hat{w}}}, \bigoplus_{-1_{\hat{w}}}, \bigoplus_{2_{\hat{w}}}\}$.

Proof Let $[a_{ij}^1], [a_{ij}^2] \in \mathrm{FSM}_{t\times v}$. For the normalized fuzzy weighted geometric mean operator, we take $[c_{ij}] = [a_{ij}^1]\bigoplus_{0_{\hat{w}}}[a_{ij}^2]$ and $[d_{ij}] = [a_{ij}^2]\bigoplus_{0_{\hat{w}}}[a_{ij}^1]$. By Definition 20, we can write $c_{ij} = (a_{ij}^1)^{\hat{w}_1}(a_{ij}^2)^{\hat{w}_2} = (a_{ij}^1)^{\frac{w_1}{w_1+w_2}}(a_{ij}^2)^{\frac{w_2}{w_1+w_2}} = (a_{ij}^2)^{\frac{w_2}{w_1+w_2}}(a_{ij}^1)^{\frac{w_1}{w_1+w_2}} = (a_{ij}^2)^{\hat{w}_2}(a_{ij}^1)^{\hat{w}_1} = d_{ij}$ for all i, j. Thus, we have $[c_{ij}] = [d_{ij}]$. This completes the proof for the normalized fuzzy weighted geometric mean operator.

In a similar way, the commutativity of the other operators can be proved.

Note In Proposition 7, when the operators of harmonic mean and normalized fuzzy weighted harmonic mean are taken, it must be born in mind that $a_{ij}^1 \neq 0$ and $a_{ij}^2 \neq 0$ for all i, j.

Definition 25 (*Power mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a power mean (PM) (or pth power mean (p-PM)) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_p [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_p [a_{ij}^2] \bigoplus_p \ldots \bigoplus_p [a_{ij}^s]$, if for all i, j

$$c_{ij} = \begin{cases} \left(\frac{1}{s}\sum_{\ell=1}^{s}\left(a_{ij}^\ell\right)^p\right)^{\frac{1}{p}}, & \text{if } p \neq 0 \\ \left(\prod_{\ell=1}^{s} a_{ij}^\ell\right)^{\frac{1}{s}}, & \text{if } p = 0 \end{cases} \tag{29}$$

Definition 26 (*Normalized fuzzy weighted power mean of fuzzy soft matrices*) Let $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a normalized fuzzy weighted power mean (NFWPM) (or normalized fuzzy weighted pth power mean (p-NFWPM)) of $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$, denoted by $[c_{ij}] = \bigoplus_{\ell=1}^{s}{}_{p_{\hat{w}}} [a_{ij}^\ell] = [a_{ij}^1] \bigoplus_{p_{\hat{w}}} [a_{ij}^2] \bigoplus_{p_{\hat{w}}} \ldots \bigoplus_{p_{\hat{w}}} [a_{ij}^s]$, if for all i, j

$$c_{ij} = \begin{cases} \left(\sum_{\ell=1}^{s}\hat{w}_\ell\left(a_{ij}^\ell\right)^p\right)^{\frac{1}{p}}, & \text{if } p \neq 0 \\ \prod_{\ell=1}^{s}\left(a_{ij}^\ell\right)^{\hat{w}_\ell}, & \text{if } p = 0 \end{cases} \tag{30}$$

where $\hat{w}_\ell = w_\ell/(w_1 + w_2 + \cdots + w_s)$ is referred to as a normalized fuzzy weight for the fuzzy soft matrix $[a_{ij}^\ell]$. Also, $w_1, w_2, \ldots, w_s \in [0, 1]$ respectively indicate the fuzzy weights for the fuzzy soft matrices $[a_{ij}^1], [a_{ij}^2], \ldots, [a_{ij}^s]$ such that $w_1 + w_2 + \cdots + w_s > 0$.

There are some special cases:

The (normalized fuzzy weighted) pth power mean of fuzzy soft matrices is considered to be

- the (normalized fuzzy weighted) harmonic mean of fuzzy soft matrices if $p = -1$.
- the (normalized fuzzy weighted) geometric mean of fuzzy soft matrices if $p = 0$.
- the (normalized fuzzy weighted) arithmetic mean of fuzzy soft matrices if $p = 1$.
- the (normalized fuzzy weighted) quadratic mean of fuzzy soft matrices if $p = 2$.

Proposition 8 *Let* $[a_{ij}^1], [a_{ij}^2], \ldots [a_{ij}^s] \in FSM_{t\times v}$ *such that and* $a_{ij}^\ell \neq 0$ *for all* $\ell = 1, 2, \ldots, s$. *Then,*

(i) $\bigoplus_{\ell=1}^{s}{}_p [a_{ij}^\ell] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_q [a_{ij}^\ell] \Leftrightarrow \bigoplus_{\ell=1}^{s}{}_{-p} [a_{ij}^\ell] \widetilde{\supseteq} \bigoplus_{\ell=1}^{s}{}_{-q} [a_{ij}^\ell]$.

(ii) $\bigoplus_{\ell=1}^{s}{}_{p_{\hat{w}}} [a_{ij}^\ell] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{q_{\hat{w}}} [a_{ij}^\ell] \Leftrightarrow \bigoplus_{\ell=1}^{s}{}_{-p_{\hat{w}}} [a_{ij}^\ell] \widetilde{\supseteq} \bigoplus_{\ell=1}^{s}{}_{-q_{\hat{w}}} [a_{ij}^\ell]$.

Proof (i) Let $[a_{ij}^1], [a_{ij}^2], \ldots [a_{ij}^s] \in \mathrm{FSM}_{t\times v}$ such that and $a_{ij}^\ell \neq 0$ for all $\ell = 1, 2, \ldots, s$.

$\Rightarrow$: Let's take $\bigoplus_{\ell=1}^{s}{}_{p}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{q}[a_{ij}^{\ell}]$. If $p = 0$ or $q = 0$ then the proof is trivial by Eqs. (27) and (28). If $p \neq 0$ and $q \neq 0$ then we write, by Definition 25,

$$\left(\frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{p}\right)^{\frac{1}{p}} \leq \left(\frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{q}\right)^{\frac{1}{q}} \tag{31}$$

and so

$$\left(\frac{1}{s}\sum_{\ell=1}^{s}\frac{1}{(a_{ij}^{\ell})^{p}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{s}\sum_{\ell=1}^{s}\frac{1}{(a_{ij}^{\ell})^{q}}\right)^{\frac{1}{q}} \tag{32}$$

When both sides of Eq. (32) are risen to power of -1, we obtain

$$\left(\frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{-p}\right)^{-\frac{1}{p}} = \left(\frac{s}{\sum_{\ell=1}^{s}\frac{1}{(a_{ij}^{\ell})^{p}}}\right)^{\frac{1}{p}}$$

$$\geq \left(\frac{s}{\sum_{\ell=1}^{s}\frac{1}{(a_{ij}^{\ell})^{q}}}\right)^{\frac{1}{q}} = \left(\frac{1}{s}\sum_{\ell=1}^{s}(a_{ij}^{\ell})^{-q}\right)^{-\frac{1}{q}}.$$

Thus, we prove that $\bigoplus_{\ell=1}^{s}{}_{-p}[a_{ij}^{\ell}] \widetilde{\supseteq} \bigoplus_{\ell=1}^{s}{}_{-q}[a_{ij}^{\ell}]$ if $\bigoplus_{\ell=1}^{s}{}_{p}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{q}[a_{ij}^{\ell}]$.

$\Leftarrow$: By using similar techniques, it can be proved that

$$\bigoplus_{\ell=1}^{s}{}_{p}[a_{ij}^{\ell}] \widetilde{\subseteq} \bigoplus_{\ell=1}^{s}{}_{q}[a_{ij}^{\ell}] \text{ if } \bigoplus_{\ell=1}^{s}{}_{-p}[a_{ij}^{\ell}] \widetilde{\supseteq} \bigoplus_{\ell=1}^{s}{}_{-q}[a_{ij}^{\ell}].$$

The proof of (ii) is similar, hence omitted.

Definition 27 (*Mid-range of fuzzy soft matrices*) Let $[a_{ij}^{1}], [a_{ij}^{2}], \ldots, [a_{ij}^{s}] \in \mathrm{FSM}_{t\times v}$. Then $[c_{ij}]$ is called a mid-range operator of $[a_{ij}^{1}], [a_{ij}^{2}], \ldots, [a_{ij}^{s}]$, denoted by $[c_{ij}] = Mid([a_{ij}^{1}], [a_{ij}^{2}], \ldots, [a_{ij}^{s}])$, if for all i, j

$$c_{ij} = \frac{1}{2}\left(\max_{\ell\in\{1,2,\ldots,s\}} a_{ij}^{\ell} + \min_{\ell\in\{1,2,\ldots,s\}} a_{ij}^{\ell}\right) \tag{33}$$

In particular, from this definition, we have $Mid([a_{ij}^{1}]) = [a_{ij}^{1}]$ when $s = 1$, and $Mid([a_{ij}^{1}], [a_{ij}^{2}]) = [a_{ij}^{1}] \bigoplus_{1} [a_{ij}^{2}]$ when $s = 2$

Example 10 Let us take the fuzzy soft matrices $[a_{ij}^{1}], [a_{ij}^{2}], [a_{ij}^{3}] \in \mathrm{FSM}_{4\times 3}$ given in Example 8. Then, we obtain

$$[c_{ij}] = Mid([a_{ij}^{1}], [a_{ij}^{2}], [a_{ij}^{3}]) = \begin{bmatrix} 0.5 & 0.55 & 0.25 \\ 0.6 & 1 & 0.55 \\ 0.05 & 0.65 & 0.25 \\ 0.35 & 0.55 & 0.4 \end{bmatrix}.$$

5 The proposed fuzzy soft multicriteria group decision making methods

In this section, we first describe the problem of multicriteria group decision making with different evaluation criterion sets. We then develop an MCGDM algorithm (Algorithm 1) that overcomes this kind of decision making problems. Furthermore, we propose an MCGDM algorithm (Algorithm 2) to handle the multicriteria group decision making with the same evaluation criterion set, which consists of only one criterion set.

5.1 Description of MCGDM problem with different evaluation criterion sets

Fan et al. (2018) describe the MCGDM problem with different evaluation criterion sets. Let's reinterpret this problem as follows.

From now on, we use the following notation to denote the sets and variables in the multicriteria group decision making with different evaluation criterion sets.

(i) $U = \{u_1, u_2, \ldots, u_t\}$ is the set of t alternatives, where u_i $(i = 1, 2, \ldots, t)$ denotes the ith alternative,
(ii) $Q = \{Q_1, Q_2, \ldots, Q_r\}$ is the set of r experts (decision makers), where Q_n $(n = 1, 2, \ldots, r)$ denotes the nth expert,
(iii) $w = (w_1, w_2, \ldots, w_r)$ is the weight vector of experts, where w_n $(n = 1, 2, \ldots, r)$ denotes fuzzy weight of expert Q_n,
(iv) $X = \{X_1, X_2, \ldots, X_m\}$ is the set of criterion sets (main criteria and related sub-criteria) that experts use to evaluate the alternatives, where X_k $(k = 1, 2, \ldots, s)$ denotes the kth criterion set,
(v) X^n $(n = 1, 2, \ldots, r)$ is the set of criterion sets concerned by the expert Q_n such that $X^n \subseteq X$ and $X^1 \cup X^2 \cup \ldots \cup X^r = X$. The set X^n is called an evaluation criterion set determined by Q_n.

If $X^n = \emptyset$ then it means that Q_n is not interested in any criteria and does not need to be included as an expert in decision analysis. Therefore, we suppose that $X^n \neq \emptyset$ in this paper.

Notation $[a_{ijk}^{(n,k)}]$ denotes the fuzzy soft matrix obtained from the evaluation of alternatives with respect to the sub-criteria in X_k by the nth expert.

5.2 The proposed methods for MCGDM

Definition 28 Assume that Q_n and w_n $(n = 1, 2, \ldots, r)$ indicate the experts and their fuzzy weights, respectively.

(a) $[a_{ijk}^{k}] = \bigoplus_{p\ n\in\aleph} [a_{ijk}^{(n,k)}]$ is called a k-aggregate pth power mean fuzzy soft matrix (briefly k-aggregate $p - PMfs$ matrix), where $\aleph = \{n : n$ signifies the subscript notation of $Q_n^{\dagger}\}$ such that $Q_n^{\dagger}$ indicates the expert who evaluates the alternatives with respect to the sub-criteria in the set X_k.
(b) $[a_{ijk}^{k}] = \bigoplus_{p_{\hat{w}}\ n\in\aleph} [a_{ijk}^{(n,k)}]$ is called a k-aggregate normalized fuzzy weighted pth power mean fuzzy soft matrix (briefly k-aggregate $p - NFWPMfs$ matrix), where $\aleph = \{n : n$ signifies the subscript notation of $Q_n^{\dagger}\}$ such that $Q_n^{\dagger}$ indicates the expert who evaluates the alternatives with respect to the sub-criteria in the set X_k.

(c) $[a_{ij_k}^k] = \bigoplus_{n\in\aleph}{}_p\, [a_{ij_k}^{(n,k)}]^{w_n}$ is called a k-aggregate pth power mean fuzzy-weighted fuzzy soft matrix (briefly k-aggregate $p - PMfwfs$ matrix), where $\aleph = \{n : n$ signifies the subscript notation of $Q_n^{\dagger}\}$ such that $Q_n^{\dagger}$ indicates the expert who evaluates the alternatives with respect to the sub-criteria in the set X_k. Also, w_n denotes the fuzzy weight for the fuzzy soft matrix $[a_{ij_k}^{(n,k)}]$.

Definition 29 Let $[c_{ip}]$ be a fuzzy soft matrix. Then, *fuzzy soft max-column function* f_M is defined as follows

$$f_M : \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{1\times v},\ f_M([c_{ip}]) = [c_{1p}^M] \tag{34}$$

where $c_{1p}^M = \max_{i\in\{1,2,\ldots,t\}} c_{ip}$. The matrix $[c_{1p}^M]$ is called a *max-column fuzzy soft matrix* of $[c_{ip}]$.

Definition 30 Let $[c_{ip}]$ be a fuzzy soft matrix. Then, *fuzzy soft min-column function* f_m is defined as follows

$$f_m : \mathrm{FSM}_{t\times v} \longrightarrow \mathrm{FSM}_{1\times v},\ f_m([c_{ip}]) = [c_{1p}^m] \tag{35}$$

where $c_{1p}^m = \min_{i\in\{1,2,\ldots,t\}} c_{ip}$. The matrix $[c_{1p}^m]$ is called a *min-column fuzzy soft matrix* of $[c_{ip}]$.

Definition 31 Let $[c_{ip}]$ be a fuzzy soft matrix. Also, let $[c_{1p}^M]$ and $[c_{1p}^m]$ be a max-column fuzzy soft matrix and a min-column fuzzy soft matrix of $[c_{ip}]$, respectively. Then $[c_{ip}^*]$ is called a normalized decision matrix of $[c_{ip}]$, where c_{ip}^* indicates the normalized value of c_{ip} and it is calculated by the following:

$$c_{ip}^* = \begin{cases} \dfrac{c_{ip} - c_{1p}^m}{c_{1p}^M - c_{1p}^m}, & \text{if } c_{1p}^M \neq c_{1p}^m \\ 0, & \text{if } c_{1p}^M = c_{1p}^m \end{cases} \tag{36}$$

Definition 32 Let $U = \{u_1, u_2, \ldots, u_t\}$ be a set of alternatives. By using the normalized decision matrix $[c_{ip}^*]_{t\times v}$, the ranking value of alternative u_i can be calculated as follows:

$$\mathrm{rank}_{[c_{ip}^*]}(u_i) = \sum_{p=1}^{v} c_{ip}^*. \tag{37}$$

Now, we construct a novel algorithm for MCGDM problem with different evaluation criterion sets.

Algorithm 1

Step 1. For $n = 1, 2, \ldots, r$, the nth expert determines his/her evaluation criterion set X^n from the set $X = \{X_1, X_2, \ldots, X_m\}$ which provides the condition $|X| > 1$.

Step 2. For $n = 1, 2, \ldots, r$, the nth expert creates the fuzzy soft matrix $[a_{ij_k}^{(n,k)}]$ for each set X_k in the evaluation criterion set X^n.

Step 3. If for each X_k there is the set $\wp^k \subseteq \{1, 2, \ldots, s\}$ with $|\wp^k| \geq 2$ that provides $X_k \in \bigcap_{n\in\wp^k} X^n$, then obtain the k-aggregate $p - NFWPMfs$ matrix $[a_{ij_k}^k]$ for each $k = 1, 2, \ldots, m$, otherwise find the k-aggregate $p - PMfwfs$ matrix $[a_{ij_k}^k]$ for each $k = 1, 2, \ldots, m$.

Table 1 The main criteria and related sub-criteria for smartphone selection

X_1 = market conditions	X_2 = physical conditions	X_3 = functionality conditions	X_4 = technical conditions
x_1^1 = price	x_1^2 = dimensions	x_1^3 = variety of applications	x_1^4 = processor speed
x_2^1 = service availability	x_2^2 = battery life	x_2^3 = operational system easiness	x_2^4 = memory capability
			x_3^4 = camera specifications

Step 4. Find $[c_{ip}^1] = \underset{k=1}{\overset{m}{\curlywedge}} [a_{ij_k}^k] = [a_{ij_1}^1] \curlywedge [a_{ij_2}^2] \curlywedge \ldots \curlywedge [a_{ij_m}^m]$ and $[c_{ip}^2] = \underset{k=1}{\overset{m}{\curlyvee}} [a_{ij_k}^k] = [a_{ij_1}^1] \curlyvee [a_{ij_2}^2] \curlyvee \ldots \curlyvee [a_{ij_m}^m]$,

Step 5. Find $[c_{ip}] = Mid([c_{ij}^1], [c_{ij}^2])$,

Step 6. Find the max-column fuzzy soft matrix $[c_{1p}^M]$ and the min-column fuzzy soft matrix $[c_{1p}^m]$.

Step 7. Construct the normalized decision matrix $[c_{ip}^*]$.

Step 8. Calculate the ranking value of alternative u_i for $i = 1, 2, \ldots, t$ and then determine optimal choice.

In Step 3 of this Algorithm, we can use any of the (normalized fuzzy weighted) pth power mean operators of fuzzy soft matrices for the various reasonably applicable decision making issues. In this study, we usually concentrate on the results of Algorithm 1 for $p = -1, 0, 1, 2$.

Let us illustrate an implementation of the above multicriteria group decision making algorithm.

Example 11 (adapted from Büyüközkan and Güleryüz (2016)) Assume that a Turkish Company aims to purchase smartphones for their white-collar employees. A decision committee consisting of three experts Q_1, Q_2 and Q_3 aims to determine the most appropriate smartphone among seven possible alternatives. These alternative smartphones can be considered as the set $U = \{u_1, u_2, \ldots, u_7\}$. Q_1 is the procurement manager who is well informed about available smartphone alternatives. Q_2 is the quality manager who concentrates on the quality of the products. Q_3 is the human resources manager of the company, who has enough information about the profile of Company's employees and their smartphone preferences. In the decision making process, the company determines the weight vector of three experts as $w = (0.9, 0.2, 0.4)$. Also, the criterion sets (main criteria and sub-criteria) to be used by experts in the evaluation of alternatives are given in Table 1.

To allow decision making process, the experts analyze the criteria and alternatives according to their interest and expertise.

Step 1 The experts Q_1, Q_2 and Q_3 determine the evaluation criterion sets as $X^1 = \{X_1, X_2\}$, $X^2 = \{X_2, X_3, X_4\}$ and $X^3 = \{X_1, X_2, X_3, X_4\}$, respectively.

Step 2 The expert Q_1 creates the following fuzzy soft matrices for each of the sets X_1 and X_2 in the evaluation criterion set X^1.

$$[a_{ij_1}^{(1,1)}] = \begin{bmatrix} 0.3 & 0.6 \\ 0.35 & 0.1 \\ 0.8 & 0.8 \\ 0.4 & 0.4 \\ 0.7 & 0.3 \\ 0.7 & 0.5 \\ 0.6 & 0.5 \end{bmatrix} \text{ and } [a_{ij_2}^{(1,2)}] = \begin{bmatrix} 0.56 & 0.2 \\ 0.12 & 0.33 \\ 0.7 & 0.6 \\ 0.6 & 0.56 \\ 0.5 & 0.7 \\ 0.6 & 0.6 \\ 0.65 & 0.65 \end{bmatrix}.$$

The expert Q_2 creates the following fuzzy soft matrices for each of the sets X_2, X_3 and X_4 in the evaluation criterion set X^2.

$$[a_{ij_2}^{(2,2)}] = \begin{bmatrix} 0.6 & 0.1 \\ 0.25 & 0.33 \\ 0.6 & 0.5 \\ 0.59 & 0.44 \\ 0.5 & 0.47 \\ 0.8 & 0.8 \\ 0.65 & 0.35 \end{bmatrix}, [a_{ij_3}^{(2,3)}] = \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 0.4 \\ 0.8 & 0.5 \\ 0.6 & 0.3 \\ 0.5 & 0.3 \\ 0.5 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \text{ and } [a_{ij_4}^{(2,4)}] = \begin{bmatrix} 0.5 & 0.4 & 0.4 \\ 0.7 & 0.7 & 0.6 \\ 0.4 & 0.4 & 0.4 \\ 0.5 & 0.55 & 0.35 \\ 0.4 & 0.3 & 0.5 \\ 0.8 & 0.7 & 0.8 \\ 0.4 & 0.4 & 0.5 \end{bmatrix}.$$

The expert Q_3 creates the following fuzzy soft matrices for each of the sets X_1, X_2, X_3 and X_4 in the evaluation criterion set X^3.

$$[a_{ij_1}^{(3,1)}] = \begin{bmatrix} 0.4 & 0.4 \\ 0.5 & 0.3 \\ 0.5 & 0.5 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.6 & 0.6 \\ 0.6 & 0.3 \end{bmatrix}, \qquad [a_{ij_2}^{(3,2)}] = \begin{bmatrix} 0.74 & 0.15 \\ 0.23 & 0.33 \\ 0.8 & 0.7 \\ 0.7 & 0.5 \\ 0.65 & 0.6 \\ 0.65 & 0.65 \\ 0.6 & 0.6 \end{bmatrix},$$

$$[a_{ij_3}^{(3,3)}] = \begin{bmatrix} 0.84 & 0.5 \\ 0.7 & 0.5 \\ 0.75 & 0.4 \\ 0.6 & 0.3 \\ 0.56 & 0.42 \\ 0.5 & 0.6 \\ 0.6 & 0.3 \end{bmatrix}, \qquad [a_{ij_4}^{(3,4)}] = \begin{bmatrix} 0.7 & 0.5 & 0.5 \\ 0.4 & 0.25 & 0.8 \\ 0.7 & 0.5 & 0.6 \\ 0.5 & 0.55 & 0.35 \\ 0.4 & 0.3 & 0.5 \\ 0.65 & 0.6 & 0.4 \\ 0.4 & 0.3 & 0.7 \end{bmatrix}.$$

Step 3 Since $X_1 \in X^1 \cap X^3$, $X_2 \in X^1 \cap X^2 \cap X^3$, $X_3 \in X^2 \cap X^3$ and $X_4 \in X^2 \cap X^3$, the k-aggregate $p - NFWPMfs$ matrices should be obtained in this step. Hence, the k-aggregate $p - NFWPMfs$ matrices for $p = 1$ (i.e. k-aggregate normalized fuzzy weighted arithmetic mean fuzzy soft matrices) are found for $k = 1, 2, 3, 4$ as follows:

$$[a^1_{ij_1}] = \bigoplus_{n\in\{1,3\}} 1_{\hat{w}}[a^{(n,1)}_{ij_1}] = \begin{bmatrix} 0.33 & 0.538 \\ 0.396 & 0.161 \\ 0.707 & 0.707 \\ 0.338 & 0.369 \\ 0.638 & 0.392 \\ 0.669 & 0.53 \\ 0.6 & 0.438 \end{bmatrix},$$

$$[a^2_{ij_2}] = \bigoplus_{n\in\{1,2,3\}} 1_{\hat{w}}[a^{(n,2)}_{ij_2}] = \begin{bmatrix} 0.613 & 0.173 \\ 0.166 & 0.33 \\ 0.713 & 0.613 \\ 0.625 & 0.528 \\ 0.54 & 0.642 \\ 0.64 & 0.64 \\ 0.636 & 0.596 \end{bmatrix},$$

$$[a^3_{ij_3}] = \bigoplus_{n\in\{2,3\}} 1_{\hat{w}}[a^{(n,3)}_{ij_3}] = \begin{bmatrix} 0.826 & 0.5 \\ 0.666 & 0.466 \\ 0.766 & 0.433 \\ 0.6 & 0.3 \\ 0.54 & 0.38 \\ 0.5 & 0.566 \\ 0.533 & 0.366 \end{bmatrix} \text{ and}$$

$$[a^4_{ij_4}] = \bigoplus_{n\in\{2,3\}} 1_{\hat{w}}[a^{(n,4)}_{ij_4}] = \begin{bmatrix} 0.633 & 0.466 & 0.466 \\ 0.5 & 0.4 & 0.733 \\ 0.6 & 0.466 & 0.533 \\ 0.5 & 0.55 & 0.35 \\ 0.4 & 0.3 & 0.5 \\ 0.7 & 0.633 & 0.533 \\ 0.4 & 0.333 & 0.633 \end{bmatrix}.$$

Step 4–7 The fuzzy soft matrix $[c_{ip}] = Mid([c^1_{ij}], [c^2_{ij}])$, the max-column fuzzy soft matrix $[c^M_{1p}]$, the min-column fuzzy soft matrix $[c^m_{1p}]$ and then the normalized decision matrix $[c^*_{ip}]$ are found as follows:

$$[c_{ip}^{1}] = \begin{bmatrix}
0.33 & 0.33 & 0.33 & 0.33 & 0.33 & 0.173 & 0.173 & 0.173 & 0.173 & 0.538 & 0.466 & 0.466 & 0.466 & 0.173 & 0.173 & 0.173 & 0.173 & 0.173 & 0.173 \\
0.166 & 0.166 & 0.166 & 0.166 & 0.166 & 0.33 & 0.33 & 0.33 & 0.33 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 & 0.161 \\
0.6 & 0.466 & 0.533 & 0.433 & 0.433 & 0.6 & 0.533 & 0.433 & 0.433 & 0.6 & 0.466 & 0.533 & 0.433 & 0.6 & 0.466 & 0.533 & 0.433 & 0.433 & 0.433 \\
0.338 & 0.338 & 0.338 & 0.3 & 0.3 & 0.338 & 0.338 & 0.3 & 0.3 & 0.369 & 0.369 & 0.35 & 0.3 & 0.369 & 0.369 & 0.35 & 0.3 & 0.3 & 0.3 \\
0.4 & 0.3 & 0.5 & 0.38 & 0.3 & 0.4 & 0.5 & 0.38 & 0.3 & 0.392 & 0.3 & 0.392 & 0.38 & 0.392 & 0.3 & 0.392 & 0.38 & 0.3 & 0.38 \\
0.5 & 0.5 & 0.5 & 0.566 & 0.533 & 0.5 & 0.566 & 0.566 & 0.566 & 0.5 & 0.5 & 0.5 & 0.53 & 0.5 & 0.5 & 0.5 & 0.53 & 0.53 & 0.53 \\
0.4 & 0.333 & 0.533 & 0.366 & 0.333 & 0.366 & 0.533 & 0.366 & 0.333 & 0.366 & 0.333 & 0.438 & 0.366 & 0.4 & 0.333 & 0.438 & 0.366 & 0.333 & 0.366
\end{bmatrix},$$

$$[c_{ip}^{2}] = \begin{bmatrix}
0.826 & 0.826 & 0.826 & 0.633 & 0.613 & 0.613 & 0.826 & 0.826 & 0.633 & 0.5 & 0.826 & 0.826 & 0.633 & 0.613 & 0.613 & 0.826 & 0.826 & 0.633 & 0.538 & 0.538 \\
0.666 & 0.666 & 0.733 & 0.5 & 0.466 & 0.733 & 0.666 & 0.733 & 0.5 & 0.466 & 0.666 & 0.733 & 0.5 & 0.466 & 0.733 & 0.666 & 0.733 & 0.5 & 0.466 & 0.733 \\
0.766 & 0.766 & 0.766 & 0.713 & 0.713 & 0.713 & 0.766 & 0.766 & 0.707 & 0.707 & 0.766 & 0.766 & 0.713 & 0.713 & 0.713 & 0.766 & 0.766 & 0.707 & 0.707 & 0.707 \\
0.625 & 0.625 & 0.625 & 0.625 & 0.625 & 0.625 & 0.6 & 0.6 & 0.528 & 0.55 & 0.625 & 0.625 & 0.625 & 0.625 & 0.625 & 0.6 & 0.6 & 0.528 & 0.55 & 0.528 \\
0.638 & 0.638 & 0.638 & 0.638 & 0.638 & 0.638 & 0.642 & 0.642 & 0.642 & 0.642 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.642 & 0.642 & 0.642 & 0.642 & 0.642 \\
0.7 & 0.669 & 0.669 & 0.7 & 0.669 & 0.669 & 0.7 & 0.669 & 0.7 & 0.669 & 0.64 & 0.64 & 0.64 & 0.7 & 0.64 & 0.64 & 0.64 & 0.7 & 0.64 & 0.64 \\
0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.636 & 0.596 & 0.596 & 0.633 & 0.596 & 0.633
\end{bmatrix},$$

$$[c_{ip}] = \begin{bmatrix}
0.578 & 0.578 & 0.578 & 0.481 & 0.471 & 0.471 & 0.499 & 0.499 & 0.403 & 0.336 & 0.336 & 0.682 & 0.646 & 0.646 & 0.566 & 0.539 & 0.539 & 0.499 & 0.499 & 0.499 & 0.403 & 0.355 & 0.355 \\
0.416 & 0.416 & 0.449 & 0.333 & 0.316 & 0.449 & 0.498 & 0.498 & 0.531 & 0.415 & 0.398 & 0.531 & 0.413 & 0.413 & 0.33 & 0.313 & 0.447 & 0.413 & 0.413 & 0.447 & 0.33 & 0.313 & 0.447 \\
0.683 & 0.616 & 0.649 & 0.573 & 0.573 & 0.573 & 0.683 & 0.616 & 0.649 & 0.57 & 0.57 & 0.57 & 0.683 & 0.616 & 0.649 & 0.573 & 0.573 & 0.573 & 0.683 & 0.616 & 0.649 & 0.57 & 0.57 \\
0.481 & 0.481 & 0.481 & 0.462 & 0.462 & 0.462 & 0.469 & 0.469 & 0.469 & 0.414 & 0.425 & 0.414 & 0.497 & 0.497 & 0.462 & 0.462 & 0.462 & 0.484 & 0.484 & 0.475 & 0.414 & 0.425 & 0.414 \\
0.519 & 0.469 & 0.569 & 0.509 & 0.469 & 0.509 & 0.521 & 0.471 & 0.571 & 0.511 & 0.471 & 0.511 & 0.466 & 0.466 & 0.46 & 0.442 & 0.446 & 0.517 & 0.471 & 0.517 & 0.511 & 0.471 & 0.511 \\
0.6 & 0.584 & 0.584 & 0.633 & 0.617 & 0.601 & 0.6 & 0.584 & 0.584 & 0.633 & 0.617 & 0.601 & 0.6 & 0.57 & 0.615 & 0.585 & 0.585 & 0.6 & 0.57 & 0.57 & 0.615 & 0.585 & 0.585 \\
0.518 & 0.484 & 0.584 & 0.501 & 0.484 & 0.501 & 0.5 & 0.466 & 0.583 & 0.483 & 0.466 & 0.499 & 0.518 & 0.484 & 0.537 & 0.501 & 0.484 & 0.501 & 0.498 & 0.464 & 0.535 & 0.481 & 0.464 & 0.499
\end{bmatrix},$$

$$[c_{1p}^{M}] = \begin{bmatrix} 0.683 & 0.616 & 0.649 & 0.633 & 0.617 & 0.601 & 0.683 & 0.616 & 0.649 & 0.633 & 0.617 & 0.601 & 0.683 & 0.646 & 0.649 & 0.615 & 0.585 & 0.585 & 0.683 & 0.616 & 0.649 & 0.615 & 0.585 & 0.585 \end{bmatrix},$$

$$[c_{1p}^{m}] = \begin{bmatrix} 0.416 & 0.416 & 0.449 & 0.333 & 0.316 & 0.449 & 0.469 & 0.466 & 0.469 & 0.403 & 0.336 & 0.336 & 0.413 & 0.413 & 0.447 & 0.33 & 0.313 & 0.447 & 0.413 & 0.413 & 0.447 & 0.33 & 0.313 & 0.355 \end{bmatrix},$$

$$[c_{ip}^{*}] = \begin{bmatrix}
0.606 & 0.81 & 0.645 & 0.493 & 0.514 & 0.144 & 0.14 & 0.22 & 0.166 & 0 & 0.735 & 0 & 0.996 & 1 & 0.985 & 0.828 & 0.83 & 0.666 & 0.318 & 0.423 & 0.257 & 0.256 & 0.154 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.135 & 0.213 & 0.344 & 0.052 & 0.22 & 0.883 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 \\
1 & 1 & 1 & 0.8 & 0.853 & 0.815 & 1 & 1 & 1 & 0.726 & 0.832 & 1 & 1 & 0.871 & 1 & 0.852 & 0.955 & 0.913 & 1 & 1 & 1 & 0.842 & 0.944 & 0.934 \\
0.243 & 0.325 & 0.16 & 0.43 & 0.485 & 0.085 & 0 & 0.02 & 0 & 0.047 & 0.316 & 0.294 & 0.311 & 0.36 & 0.198 & 0.463 & 0.547 & 0.108 & 0.262 & 0.349 & 0.138 & 0.294 & 0.411 & 0.256 \\
0.385 & 0.265 & 0.6 & 0.586 & 0.508 & 0.394 & 0.242 & 0.033 & 0.566 & 0.469 & 0.48 & 0.66 & 0.196 & 0.03 & 0.094 & 0.456 & 0.393 & 0.094 & 0.385 & 0.285 & 0.346 & 0.635 & 1 & 0.678 \\
0.689 & 0.84 & 0.675 & 1 & 1 & 1 & 0.612 & 0.786 & 0.638 & 1 & 1 & 1 & 0.692 & 0.673 & 0.608 & 1 & 1 & 1 & 0.692 & 0.773 & 0.608 & 1 & 1 & 1 \\
0.382 & 0.34 & 0.675 & 0.56 & 0.558 & 0.342 & 0.144 & 0 & 0.633 & 0.347 & 0.462 & 0.615 & 0.388 & 0.304 & 0.445 & 0.6 & 0.628 & 0.391 & 0.314 & 0.251 & 0.435 & 0.529 & 0.555 & 0.626
\end{bmatrix}.$$

Table 2 The ranking values of alternatives (smartphones) for $p = 1$

i	1	2	3	4	5	6	7
$rank(u_i)$	10.451	2.099	22.22	6.102	9.36	20.286	10.524

Table 3 The ranking order for $p = -1, 0, 2$

p	Ranking order
-1	$u_3 \succ u_6 \succ u_1 \succ u_7 \succ u_5 \succ u_4 \succ u_2$
0	$u_3 \succ u_6 \succ u_7 \succ u_5 \succ u_4 \succ u_1 \succ u_2$
2	$u_3 \succ u_6 \succ u_7 \succ u_5 \succ u_4 \succ u_1 \succ u_2$

Table 4 The ranking order for $p = -1, 0, 1, 2$

p	Ranking order
-1	$u_6 \succ u_3 \succ u_1 \succ u_7 \succ u_5 \succ u_4 \succ u_2$
0	$u_6 \succ u_3 \succ u_1 \succ u_7 \succ u_5 \succ u_4 \succ u_2$
1	$u_6 \succ u_3 \succ u_1 \succ u_7 \succ u_5 \succ u_4 \succ u_2$
2	$u_6 \succ u_3 \succ u_7 \succ u_1 \succ u_5 \succ u_4 \succ u_2$

Step 8 The ranking values of alternatives u_i for i=1, 2, ..., 8 are calculated as in Table 2.

Then, the ranking order of alternatives (smartphones) as follows:

$$u_3 \succ u_6 \succ u_7 \succ u_1 \succ u_5 \succ u_4 \succ u_2.$$

Hence, u_3 is the most suitable smartphone to be purchased by a Turkish company for the white-collar employees.

In Step 3 of this algorithm, if we take the k-aggregate $p - NFWPMfs$ matrices for $p = -1, 0, 2$ then we have Table 3.

Note In Example 11, if it is taken $w = (1, 1, 1)$ then the ranking order of alternatives (smartphones) is obtained for each $p = -1, 0, 1, 2$ as in Table 4.

Example 12 Consider the MCGDM problem given in Example 11.

Step 1 The experts Q_1, Q_2 and Q_3 determine the evaluation criterion sets as $X^1 = \{X_1, X_2\}$, $X^2 = \{X_2, X_3, X_4\}$ and $X^3 = \{X_1, X_2, X_3\}$, respectively.

Step 2 The expert Q_1 creates the fuzzy soft matrices $[a_{ij_1}^{(1,1)}]$ and $[a_{ij_1}^{(1,2)}]$ in Example 11. The expert Q_2 creates the fuzzy soft matrices $[a_{ij_1}^{(2,2)}]$, $[a_{ij_1}^{(2,3)}]$ and $[a_{ij_1}^{(2,4)}]$ in Example 11. The expert Q_3 creates the fuzzy soft matrices $[a_{ij_1}^{(3,1)}]$, $[a_{ij_1}^{(3,2)}]$ and $[a_{ij_1}^{(3,3)}]$ in Example 11.

Step 3 Since $X_4 \in X^2$, the k-aggregate $p - PMfwfs$ matrices should be obtained in this step. Hence, the k-aggregate $p - PMfwfs$ matrices for $p = 1$ (i.e. k-aggregate arithmetic mean fuzzy-weighted fuzzy soft matrices) are found for $k = 1, 2, 3, 4$ as follows:

$$[a_{ij_1}^{1}] = \bigoplus_{n\in\{1,3\}}{}_1[a_{ij_1}^{(n,1)}]^{w_n} = \begin{bmatrix} 0.215 & 0.35 \\ 0.258 & 0.105 \\ 0.46 & 0.46 \\ 0.22 & 0.24 \\ 0.415 & 0.255 \\ 0.435 & 0.345 \\ 0.39 & 0.285 \end{bmatrix},$$

Table 5 The ranking values of smartphones for $p = 1$

i	1	2	3	4	5	6	7
$rank(u_i)$	8.527	1.692	21.907	6.929	12.449	21.96	12.375

$$[a^2_{ij_2}] = \bigoplus_{n\in\{1,2,3\}}{}_1 [a^{(n,2)}_{ij_2}]^{w_n} = \begin{bmatrix} 0.307 & 0.087 \\ 0.083 & 0.165 \\ 0.357 & 0.307 \\ 0.312 & 0.264 \\ 0.27 & 0.321 \\ 0.32 & 0.32 \\ 0.318 & 0.298 \end{bmatrix},$$

$$[a^3_{ij_3}] = \bigoplus_{n\in\{2,3\}}{}_1 [a^{(n,3)}_{ij_3}]^{w_n} = \begin{bmatrix} 0.248 & 0.15 \\ 0.2 & 0.14 \\ 0.23 & 0.13 \\ 0.18 & 0.09 \\ 0.162 & 0.114 \\ 0.15 & 0.17 \\ 0.16 & 0.11 \end{bmatrix} \text{ and}$$

$$[a^4_{ij_4}] = [a^{(2,4)}_{ij_4}]^{w_2} = \begin{bmatrix} 0.1 & 0.08 & 0.08 \\ 0.14 & 0.14 & 0.12 \\ 0.08 & 0.08 & 0.08 \\ 0.1 & 0.11 & 0.07 \\ 0.08 & 0.06 & 0.1 \\ 0.16 & 0.14 & 0.16 \\ 0.08 & 0.08 & 0.1 \end{bmatrix}.$$

Step 4–7 The fuzzy soft matrix $[c_{ip}] = Mid([c^1_{ij}], [c^2_{ij}])$, the max-column fuzzy soft matrix $[c^M_{1p}]$, the min-column fuzzy soft matrix $[c^m_{1p}]$ and the normalized decision matrix $[c^*_{ip}]$ can be obtained by using the Scilab codes in Appendix. So they are omitted.
Step 8 The ranking values of alternatives (smartphones) for the k-aggregate $p - PMfwfs$ matrices ($p = 1$) are calculated as in Table 5.

Then, the ranking order of alternatives (smartphones) as follows:

$$u_6 \succ u_3 \succ u_5 \succ u_7 \succ u_1 \succ u_4 \succ u_2.$$

Thus, we say that u_6 is the most suitable smartphone to be purchased by Turkish company for the white collar employees.

Comment In Step 3 of Algorithm 1, our aim is to aggregate the fuzzy soft matrices generated by using the same criterion set with specific technical approaches. According to nature of the decision making problem, we can aggregate the fuzzy soft matrices for each criterion set by using the same pth power mean operator or different pth power mean operators. For instance, in Example 11, it can be taken as $[a^1_{ij_1}] = \bigoplus_{n\in\{1,3\}}{}_{1_{\hat{w}}} [a^{(n,1)}_{ij_1}]$, $[a^2_{ij_2}] = \bigoplus_{n\in\{1,2,3\}}{}_{2_{\hat{w}}} [a^{(n,2)}_{ij_2}]$, $[a^3_{ij_3}] = \bigoplus_{n\in\{2,3\}}{}_{0_{\hat{w}}} [a^{(n,3)}_{ij_3}]$ and $[a^4_{ij_4}] = \bigoplus_{n\in\{2,3\}}{}_{1_{\hat{w}}} [a^{(n,4)}_{ij_4}]$. Propositions 3 and 4 can be considered when selecting the appropriate mean operator(s) in the decision making process.

Now, we propose an algorithm for MCGDM problem with evaluation criterion sets which consists of the same criterion set. In other words, in Algorithm 1, if $X = \{X_1\}$ i.e. $|X| = 1$ then we have the following algorithm.

Algorithm 2

Step 1. For $n = 1, 2, \ldots, r$, the nth expert has his/her evaluation criterion set $X^n = X_1$. (this is a must since $X^n \neq \emptyset$)
Step 2. For $n = 1, 2, \ldots, r$, the nth expert creates the fuzzy soft matrix $[a^n_{ij_n}]$ for the evaluation criterion set X^n.
Step 3. If there are fuzzy weights of experts then find the fuzzy-weighted fuzzy soft matrix $[a^n_{ij_n}]_{w_n}$ of nth expert for each $n = 1, 2, \ldots, r$, otherwise go to Step 4.
Step 4. Find $[c^1_{ip}] = \overset{r}{\underset{n=1}{\curlywedge}} [a^n_{ij_n}] = [a^1_{ij_1}] \curlywedge [a^2_{ij_1}] \curlywedge \ldots \curlywedge [a^r_{ij_r}]$ and $[c^2_{ip}] = \overset{r}{\underset{n=1}{\curlyvee}} [a^n_{ij_n}] = [a^1_{ij_1}] \curlyvee [a^2_{ij_2}] \curlyvee \ldots \curlyvee [a^r_{ij_r}]$.
Step 5. Find $[c_{ip}] = Mid([c^1_{ij}], [c^2_{ij}])$.
Step 6. Find the max-column fuzzy soft matrix $[c^M_{1p}]$ and the min-column fuzzy soft matrix $[c^m_{1p}]$.
Step 7. Construct the normalized decision matrix $[c^*_{ip}]$.
Step 8. Calculate the ranking value of alternative u_i for $i = 1, 2, \ldots, t$ and then determine optimal choice.

Example 13 Assume that a company wants to give a specific job to the best possible person who can fulfill the requirements of the job. Thus, the company encounters a decision problem to find the candidate who best suits the requirements of the job. The company assigns two decision makers to decide according to the following data.

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the crisp set of five persons for the job, and $X_1 = \{x^1_1 = enterprising,\ x^1_2 = confident,\ x^1_3 = well-disciplined,\ x^1_4 = willing\ to\ take\ risks\}$ be the corresponding parameters.

After the decision makers evaluate the alternatives with respect to the corresponding parameters, the decision making process proceeds step by step.

Step 1 The first and second decision makers determine the evaluation criterion sets as $X^1 = \{X_1\}$ and $X^2 = \{X_1\}$, respectively.
Step 2 The first and second decision makers create the following fuzzy soft matrices considering the evaluation criterion sets, respectively.

$$[a^1_{ij_1}] = \begin{bmatrix} 0.2 & 0.4 & 0.4 & 1 \\ 0.7 & 0.3 & 0.1 & 1 \\ 0.1 & 0.5 & 0.6 & 0 \\ 0.4 & 0.8 & 0.3 & 0.2 \\ 0.5 & 0.8 & 0.5 & 0.4 \end{bmatrix} \quad \text{and} \quad [a^2_{ij_2}] = \begin{bmatrix} 0.3 & 0.2 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.5 & 0.5 & 0.1 \\ 0.2 & 0.7 & 0.3 & 0.7 \\ 0.5 & 0.9 & 0.5 & 0.5 \end{bmatrix}.$$

Step 4 Then, the decision makers have

$$[c^1_{ip}] = \overset{2}{\underset{n=1}{\curlywedge}} [a^n_{ij_n}] = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.4 & 0.4 & 0.3 & 0.2 & 0.4 & 0.4 & 0.3 & 0.2 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.5 & 0.5 & 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0.6 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.5 & 0.1 & 0.1 & 0.5 & 0.5 & 0.1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.4 & 0.2 & 0.7 & 0.3 & 0.7 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.8 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix},$$

Table 6 The ranking values of alternatives (candidates)

i	1	2	3	4	5
$rank(u_i)$	8.551	8.609	3.112	7.888	12.935

$$[c_{ip}^2] = \bigvee_{n=1}^{2} [a_{ij_n}^n] = \begin{bmatrix} 0.3 & 0.2 & 0.8 & 0.5 & 0.4 & 0.4 & 0.8 & 0.5 & 0.4 & 0.4 & 0.8 & 0.5 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.6 & 0.4 & 0.5 & 0.5 & 0.6 & 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.1 & 0.5 & 0.5 & 0.1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.6 & 0.6 & 0.6 & 0.6 & 0.1 & 0.5 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.8 & 0.8 & 0.8 & 0.8 & 0.3 & 0.7 & 0.3 & 0.7 & 0.2 & 0.7 & 0.3 & 0.7 \\ 0.5 & 0.9 & 0.5 & 0.5 & 0.8 & 0.9 & 0.8 & 0.8 & 0.5 & 0.9 & 0.5 & 0.5 & 0.5 & 0.9 & 0.5 & 0.5 \end{bmatrix}.$$

Step 5 They obtain the following fuzzy soft matrix by using the mid-range operator.

$$[c_{ip}] = Mid([c_{ij}^1], [c_{ij}^2])$$

$$= \begin{bmatrix} 0.25 & 0.2 & 0.5 & 0.35 & 0.35 & 0.3 & 0.6 & 0.45 & 0.35 & 0.3 & 0.6 & 0.45 & 0.65 & 0.6 & 0.9 & 0.75 \\ 0.65 & 0.55 & 0.6 & 0.6 & 0.45 & 0.35 & 0.4 & 0.4 & 0.35 & 0.25 & 0.3 & 0.3 & 0.8 & 0.7 & 0.75 & 0.75 \\ 0.1 & 0.3 & 0.3 & 0.1 & 0.3 & 0.5 & 0.5 & 0.3 & 0.35 & 0.55 & 0.55 & 0.35 & 0.05 & 0.25 & 0.25 & 0.05 \\ 0.3 & 0.55 & 0.35 & 0.55 & 0.5 & 0.75 & 0.55 & 0.75 & 0.25 & 0.5 & 0.3 & 0.5 & 0.2 & 0.45 & 0.25 & 0.45 \\ 0.5 & 0.7 & 0.5 & 0.5 & 0.65 & 0.85 & 0.65 & 0.65 & 0.5 & 0.7 & 0.5 & 0.5 & 0.45 & 0.65 & 0.45 & 0.45 \end{bmatrix}.$$

Step 6 Then, they find the max-column fuzzy soft matrix $[c_{1p}^M]$ as

$$[c_{1p}^M] = \begin{bmatrix} 0.65 & 0.7 & 0.6 & 0.6 & 0.65 & 0.85 & 0.65 & 0.75 & 0.5 & 0.7 & 0.6 & 0.5 & 0.8 & 0.7 & 0.9 & 0.75 \end{bmatrix}$$

and the min-column fuzzy soft matrix $[c_{1p}^m]$ as

$$[c_{1p}^m] = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.25 & 0.25 & 0.3 & 0.3 & 0.05 & 0.25 & 0.25 & 0.05 \end{bmatrix}.$$

Step 7 Then, they create the normalized decision matrix as follows:

$$[c_{ip}^*] = \begin{bmatrix} 0.272 & 0 & 0.666 & 0.5 & 0.142 & 0 & 0.8 & 0.333 & 0.4 & 0.111 & 1 & 0.75 & 0.8 & 0.777 & 1 & 1 \\ 1 & 0.7 & 1 & 1 & 0.428 & 0.09 & 0 & 0.222 & 0.4 & 0 & 0 & 0 & 1 & 1 & 0.769 & 1 \\ 0 & 0.2 & 0 & 0 & 0 & 0.363 & 0.4 & 0 & 0.4 & 0.666 & 0.833 & 0.25 & 0 & 0 & 0 & 0 \\ 0.363 & 0.7 & 0.166 & 0.9 & 0.571 & 0.818 & 0.6 & 1 & 0 & 0.555 & 0 & 1 & 0.2 & 0.444 & 0 & 0.571 \\ 0.727 & 1 & 0.666 & 0.8 & 1 & 1 & 1 & 0.777 & 1 & 1 & 0.666 & 1 & 0.533 & 0.888 & 0.307 & 0.571 \end{bmatrix}.$$

Step 8 The ranking values of alternatives u_i for $i = 1, 2, 3, 4, 5$ are calculated as in Table 6.

Then, the ranking order of alternatives (candidates) as follows:

$$u_5 \succ u_2 \succ u_1 \succ u_4 \succ u_3.$$

Thus, u_5 is the best possible person (candidate) who can fulfill the requirements of the job.

Note The ranking order of alternatives for decision making problem in Example 13 is found

(i) $u_5 \succ u_2 \succ u_1 = u_4 \succ u_3$ by using the fuzzy soft max–min decision method in Cagman and Enginoglu (2012).
(ii) $u_5 \succ u_1 = u_4 \succ u_2 \succ u_3$ by using the fuzzy soft min–max decision method in Inthumathi et al. (2017).

Considering the fuzzy values in the matrices $[a_{ij_1}^1]$ and $[a_{ij_2}^2]$, it can be said that the result of our algorithm for the decision making problem in Example 13 is more convincing than the others. Since the fuzzy values of u_1 and u_4 are not equal in a matched way, there may be a factor to eliminate equality in the ranking order of these two alternatives. Our algorithm is constructed by considering many factors in different aspects.

Comparison Let us present the comparison of our Algorithm 2 with some of the existing decision making algorithms based on the fuzzy soft sets. (see Table 7).

Discussion on comparison In Çağman and Enginoğlu (2010a), the authors proposed the soft max–min, min–max, min–min and max–max decision making models. Atagün et al. (2018) argued that these models have some deficiencies and then introduced the soft distributive max–min, min–max, min–min and max–max decision making models improving these models. We assert that these improvements of Atagün et al. (2018) for the soft decision making models given in Çağman and Enginoğlu (2010a) should be adapted to the fuzzy soft max–min, min–max, min–min and max–max decision making models introduced in Cagman and Enginoglu (2012). Once these improvements have been made, it is seen that the optimal choice is the same as Algorithm 2 when Example 5.1 (in No. 1) is resolved again. Examples 4.3 (in No. 2) and Example 4.4 (in No. 3) can be interpreted similarly. Even with simple calculations, it can be easily seen that $u_2 \succ u_1 \succ u_4$ in Example 4.3 and $u_4 \succ u_5$ in Example 4.4.

For Example (in No. 4), the result of the algorithm in Basu et al. (2012a) coincides with the result of our Algorithm 2. However, the optimal choice obtained by employing algorithm of Basu et al. (2012a) is u_1 if two fuzzy soft sets are $\Gamma_{A_1} = \{(e_1, \{0.99/u_1, 0.98/u_2\}), (e_2, \emptyset)\}$ and $\Gamma_{A_2} = \{(e_1, \emptyset), (e_2, \{0.98/u_2\})\}$, where $U = \{u_1, u_2\}$ and $A_1 = A_2 = X = \{e_1, e_2\}$. Our Algorithm 2 provides u_2 as an optimal choice. It is already obvious that the optimal choice will be u_2.

Renukadevi and Sangeetha (2016) argued that the solution obtained using the algorithm in Borah et al. (2012) for Example 4.1 (in No. 6) is incorrect. Therefore, using the algorithm in Borah et al. (2012), the optimal selection cannot be achieved for some decision making problem based on fuzzy soft sets. Moreover, they claimed that a reasonable optimal choice should be u_2. This coincides with the result of Algorithm 2.

Each soft set is a fuzzy soft set. Relatedly, we compare the results obtained through the soft decision making methods (in No. 8–No. 12) with the results of our algorithm. While the previous soft decision making methods focus on obtaining an optimal alternative (optimum set), Algorithm 2 supplies not only the optimum choice but also the ranking order for the selection of alternatives. This is important in decision making. The remainder of Table 7 can be interpreted similarly. Consequently, we say that all of these support the usefulness and efficiency of our algorithm.

6 Conclusion

In this study, we achieved the generalizations of some products of fuzzy soft matrices. Thus, we make possible to use these fuzzy soft matrix products in the group decision making. Also, we created the mean operators for fuzzy soft matrices and introduced certain properties and relations. Thereafter, we presented the MCGDM algorithms using newly defined products and operators to deal with uncertainties in real world problems.

It is worth mentioning that these new products and operators will develop a perspective for future research in many practical areas such as group decision making, pattern recognition, intelligent systems, simulation data validation and clustering analysis. Hereby, we think that this study which generally aims to contribute to the fuzzy soft matrices and their applications will engage the attention of many authors in the near future.

Table 7 The comparison results of our Algorithm 2 with some fuzzy soft decision making methods

No.	Problem	The used decision method	Result of the used DM	Result of our Algorithm 2
1.	Example 5.1 (Cagman and Enginoglu 2012)	Max-min method in Cagman and Enginoglu (2012)	$u_1 > u_3$	$u_3 \succ u_1 \succ u_2 \succ u_5 \succ u_4$
2.	Example 4.3 (Inthumathi et al. 2017)	Min-max method in Inthumathi et al. (2017)	$u_4 > u_2 > u_1$	$u_2 \succ u_1 \succ u_4 \succ u_5 \succ u_3$
3.	Example 4.4 (Inthumathi et al. 2017)	Min-max method in Inthumathi et al. (2017)	$u_1 > u_5 > u_4$	$u_1 \succ u_4 \succ u_5 \succ u_2 \succ u_3$
4.	Example (pp. 396) (Basu et al. 2012a)	Algorithm in Basu et al. (2012a)	$P_1 > P_2 > P_3$	$P_1 \succ P_2 \succ P_3$
5.	Case Study (Das and Borgohain 2012)	Algorithm in Das and Borgohain (2012)	$c_3 > c_2 = c_1$	$c_3 \succ c_2 \succ c_1$
6.	Example 4.1 (Renukadevi and Sangeetha 2016)	Algorithm in Borah et al. (2012)	$u_1 > u_3 > u_2$	$u_2 \succ u_1 \succ u_3$
7.	Example 7.3 (Renukadevi and Sangeetha 2016)	Algorithm (pp. 53) in Renukadevi and Sangeetha (2016)	$\{u_1\}$	$u_1 \succ u_2 = u_3$
8.	Example 5.1 (Atagün et al. 2018)	Algorithm in Atagün et al. (2018)	$\{u_1, u_3\}$	$u_3 \succ u_1 \succ u_4 \succ u_5 \succ u_2$
9.	Example 5.1 (Basu et al. 2012b)	Algorithm in Basu et al. (2012b)	$\{d_2, d_3\}$	$d_3 \succ d_2 \succ d_1 \succ d_4$
10.	Example 5.2 (Basu et al. 2012b)	Algorithm in Basu et al. (2012b)	$\{f_1\}$	$f_1 \succ f_4 \succ f_2 \succ f_3$
11.	Example 31 (Kharal 2014)	Conjecture 30 in Kharal (2014)	$\{u_3\}$	$u_3 \succ u_2 \succ u_4 \succ u_5 \succ u_1$
12.	Example 3.10 (Vijayabalaji and Ramesh 2013)	Algorithm in Vijayabalaji and Ramesh (2013)	$\{m_3\}$	$m_3 \succ m_2 \succ m_5 \succ m_1 \succ m_4$

Appendix

Scilab codes

We supply the following Scilab codes, which provide great convenience for the steps in the above fuzzy soft multicriteria group decision making algorithms (Algorithms 1 and 2).

And product ($\wedge$)

For And product:

```
function x=andproduct(a1,a2)
[t,v1]=size(a1);
[t,v2]=size(a2);
x=zeros(t,v1*v2);
 for i=1:t
  for j1=1:v1
   for j2=1:v2
   p=(j1-1)*v2+j2;
   x(i,p)=min(a1(i,j1),a2(i,j2));
   end
  end
 end
endfunction
```

For multi-And-product:

```
function c1=multiandprod(varargin)
r=argn(2);
x=andproduct(varargin(1),varargin(2));
 for i=3:r
 x=andproduct(x,varargin(i));
 end
c1=x
endfunction
```

Or product ($\vee$)

For Or-product:

```
function y=orproduct(a1,a2)
[t,v1]=size(a1);
[t,v2]=size(a2);
y=zeros(t,v1*v2);
 for i=1:t
  for j1=1:v1
   for j2=1:v2
   p=(j1-1)*v2+j2;
   y(i,p)=max(a1(i,j1),a2(i,j2));
   end
  end
 end
endfunction
```

For multi-Or-product:

```
function c2=multiorprod(varargin)
r=argn(2);
y=orproduct(varargin(1),varargin(2));
 for i=3:r
 y=orproduct(y,varargin(i));
 end
c2=y
endfunction
```

Mid-range of fuzzy soft matrices
Max-column fuzzy soft matrix
Min-column fuzzy soft matrix

For $[c_{ip}] = Mid([c_{ij}^1], [c_{ij}^2])$:

```
function c=Mid(c1,c2)
k=(c1+c2)/2;
c=k
endfunction
```

For max-column fuzzy soft matrix $[c_{ip}^M]$:

```
function cM=maxcolumn(c)
M=max(c,'r');
cM=M
endfunction
```

For min-column fuzzy soft matrix $[c_{ip}^m]$:

```
function cm=mincolumn(c)
m=min(c,'r');
cm=m
endfunction
```

Normalized decision matrix
Ranking values of alternatives

For normalized decision matrix $[c_{ip}^*]$:

```
function C=decisionmatrix(c,cM,cm)
[t,v]=size(c);
C=zeros(t,v);
 for p=1:v
  for i=1:t
   if cM(1,p)==cm(1,p)
      C(i,p)==0;
   else
 C(i,p)=(c(i,p)-cm(1,p))/(cM(1,p)-cm(1,p));
   end
  end
 end
endfunction
```

For $rank(u_i)$:

```
function R=rankingvalue(C)
r=sum(C,'c');
R=r
endfunction
```

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